We are looking for a method to measure polarisation encoded pairs of photonic qubits in the Bell-basis:

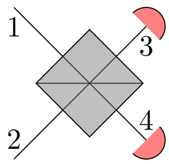
$$\begin{aligned} \left|\Psi^{\pm}\right\rangle &= \frac{1}{\sqrt{2}}\left[|H\rangle_1 |V\rangle_2 \pm |V\rangle_1 |H\rangle_2\right] \\ \left|\Phi^{\pm}\right\rangle &= \frac{1}{\sqrt{2}}\left[|H\rangle_1 |H\rangle_2 \pm |V\rangle_1 |V\rangle_2\right]. \end{aligned} \tag{1}$$

It has been shown that it is impossible to perform a complete Bell-state measurement of entangled photon pairs using only linear optics, if they are only entangled in a single degree of freedom.

Exercise 1. *Houng-Ou-Mandel interferometer*

The state of a photonic mode μ can be represented in second quantised form using bosonic operators $\hat{b}_\mu^\dagger$ that satisfy the usual commutation relations (distinct modes commute). Note that different polarisations behave as distinct modes. For example the state $|H\rangle_1 |V\rangle_2$ is then written as $\hat{b}_{1H}^\dagger \hat{b}_{2V}^\dagger |0\rangle$, where $|0\rangle$ is the electromagnetic vacuum.

The input modes 1 and 2 and the output modes 3 and 4 of a 50:50 beam splitter are connected via the boundary conditions of Maxwell's Equations:



$$\begin{aligned} \hat{b}_{1j}^\dagger &= \frac{1}{\sqrt{2}}\left(\hat{b}_{3j}^\dagger + \hat{b}_{4j}^\dagger\right) \\ \hat{b}_{2j}^\dagger &= \frac{1}{\sqrt{2}}\left(\hat{b}_{3j}^\dagger - \hat{b}_{4j}^\dagger\right) \end{aligned} \quad j = H, V \tag{2}$$

(a) Which of the four Bell states can you detect from the signal on the two detectors on ports 3 and 4, if each of the two photons of a maximally entangled pair is sent onto one input port?

Hint: The detectors cannot distinguish polarisation, nor can they count the photons. They just click if there was at least one photon, otherwise they remain silent.

(b) Given additional detectors and polarising beam splitters that transmit one polarisation and reflect the other, how many Bell states could you detect unambiguously?

Exercise 2. *Hyperentanglement-assisted deterministic Bell-measurement*

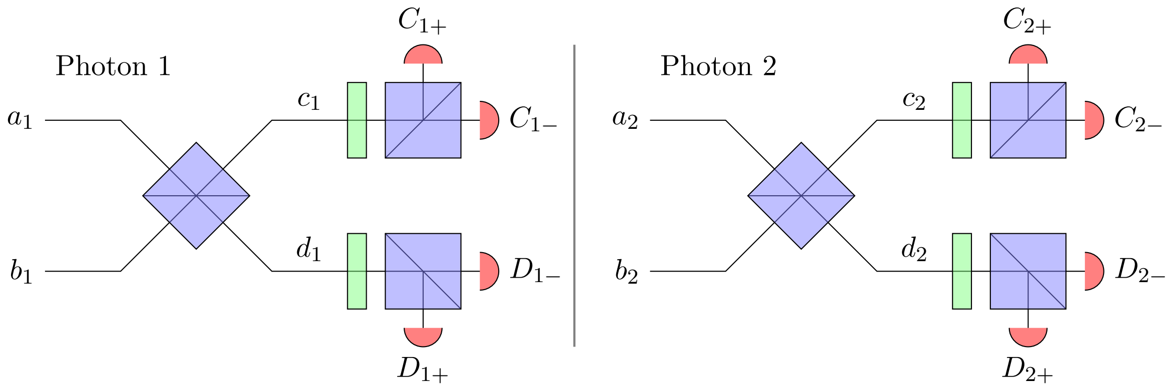
If a photon pair is entangled in more than one degree of freedom, one speaks of hyperentangled photons. Such photons are generated for example by spontaneous downconversion in nonlinear crystals. Here, we look at photon pairs simultaneously entangled in polarisation and mode.

Besides their polarisation, each photon furthermore has the option to be either in mode a or b, and besides their joint polarisation state, they shall be entangled in their mode degree of freedom. Precisely, the two-photon state shall be

$$|\Pi\rangle \otimes \left[|a\rangle_1 |b\rangle_2 + |b\rangle_1 |a\rangle_2\right], \tag{3}$$

where $|\Pi\rangle$ is any two-photon polarisation state that we want to measure in the Bell-basis.

Each photon is analysed individually using the following scheme:



Here, blue squares represent polarising beam splitters (PBS) that transmit photons in $|V\rangle$ and reflect photons in $|H\rangle$ polarisation. Green rectangles represent half-wave plates (HWP) oriented at $\pi/8$ angle, such that $|H\rangle \to \frac{1}{\sqrt{2}}[|H\rangle + |V\rangle]$ and $|V\rangle \to \frac{1}{\sqrt{2}}[|H\rangle - |V\rangle]$. Red half circles represent single-photon counters.

(a) How does the initial PBS in each analyser act on the photon's polarisation and mode degrees of freedom? If you would consider those two as separate qubits, how would you describe the operation performed by the PBS?

(b) If you would place detectors at c_1, d_1, c_2 and d_2 instead of the HWP and further PBS, how could you partition the input states based on the detector signals?

(c) Consider a single photon in mode c_1. What polarisation state in mode c_1 is identified by a click in detector C_{1+}, and by a click in C_{1-}?

(d) If you would input polarisation entangled photons at c_1 and c_2 (skipping the initial PBS), how could you partition the Bell states from the detector signal now?

(e) Would this scheme allow for deterministic entanglement swapping of a polarisation qubit?

Hint: Since each photon is analysed separately from the other, there will only ever be one photon, and we don't need to work in second quantisation.