Exercise 1. ***Electrostatic potential from a dipole***

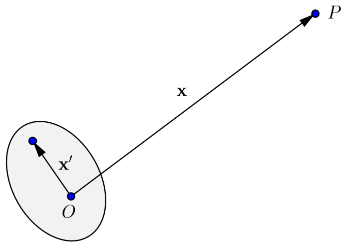


Figure 1: Potential at large distance from the charge distribution

1. The general expression for the potential is given by,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'. \tag{1}$$

Show that at large distances from the charge distribution ($|\mathbf{x}| \gg |\mathbf{x}'|$), as it is shown in figure 1, the potential can be approximated by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\Big[\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + \dots\Big]. \tag{2}$$

Where $r = |\mathbf{x}|$, q is the total charge and $\mathbf{p}$ is the electric dipole moment, defined by

$$\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')d^3x' \tag{3}$$

Solution. The solution for the potential is given by making a Taylor expansion of $1/|\mathbf{x}-\mathbf{x}'|$, i.e.

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3} + \dots \tag{S.1}$$

The final expression then follow directly.

2. Calculate the potential of the following configurations (see figure 2 for (a) and (b) and figure 3 for (c).) by applying (2),

   (a) Two identical point charges with opposite sign at distance d.

   Solution. We choose the origin to be at the place of the negative charge, then the dipole is simple given by

$$\mathbf{p} = q\,d\,\mathbf{e}_d \tag{S.2}$$

   where $\mathbf{e}_d$ is the unit vector pointing from q^- to q^+. The potential Φ follows directly.

   (b) The potential given by a molecule of water ($d_{OH} = 0,96.10^{-10}m$, $\theta = 104,5$, $Q_{O^-} = -0,66\,e$ and $Q_{H^+} = 0,33\,e$)

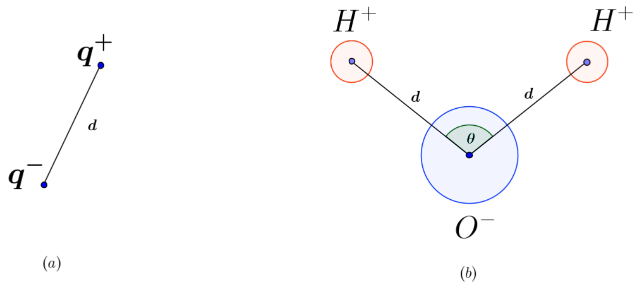


Figure 2: Different examples of dipoles

Solution. By symmetry is the dipole in the direction from the oxygen atom to the middle between the two hydrogen charges. We can then calculate directly the vertical component of $\mathbf{p}$, again we choose the coordinate system such that the origin is placed on the oxygen charge,

$$p_z = 2d\cos(\frac{\theta}{2})Q_{H^+} = 2 \cdot 0,96.10^{-10}m \cdot \cos(\frac{\frac{104,5\cdot\pi}{180}}{2}) \cdot 0,33 \cdot 1,6.10^{-19}C = 6,2.10^{-30}Cm \tag{S.3}$$

(c) Four charges forming two dipoles, according to figure 3. For which angles θ_1 and θ_2 and distances $d_1 = |\mathbf{d}_1|$ and $d_2 = |\mathbf{d}_2|$ is the total dipole moment vanishing? How are the dipoles disposed in that case?

Solution. We choose the origin to be at the place of the positive charge in the left. Then we have

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \tag{S.4}$$

$$= -qd_1\begin{pmatrix}\cos(\theta_1)\\ \sin(\theta_1)\end{pmatrix} + \begin{pmatrix}rq\\ 0\end{pmatrix} + (-q)\begin{pmatrix}r - d_2\cos(\pi-\theta_2)\\ d_2\sin(\pi-\theta_2)\end{pmatrix} \tag{S.5}$$

$$= -qd_1\begin{pmatrix}\cos(\theta_1)\\ \sin(\theta_1)\end{pmatrix} - qd_2\begin{pmatrix}\cos(\theta_2)\\ \sin(\theta_2)\end{pmatrix} \tag{S.6}$$

We see that if $\theta_1 = \pi/2$, $\theta_2 = (3\pi)/2$ and $d_1 = d_2$, then the total dipole moment vanishes. This correspond also to dispose the four charges on a square.

Exercise 2. *Van der Waals forces between 2 dipoles*

In this exercise we will see how the van der Waals forces between two molecules or atoms emerge. To this end, consider the situation depicted in the Figure 3.

Let $\boldsymbol{r}$ be the vector connecting the 2 positive charges (of the charge $+q$ each), and $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ the vectors connecting each positive charge with a negative charge $(-q)$ it forms a dipole with.

1. To start with, write down the expression for the total energy of the system. In total you should get the contribution from 6 terms. Make sure that each of them comes with a correct sign.

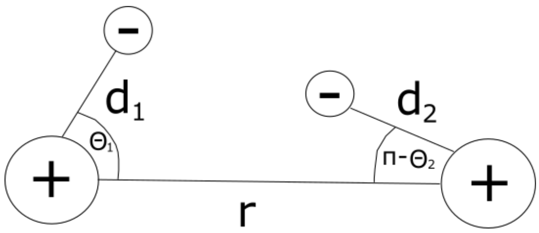


Figure 3: The system consisting of 2 dipoles.

Solution. The following terms contribute:

$$4\pi\epsilon_0 U = q^2\left(\frac{1}{|\mathbf{r}|} + \frac{1}{|\boldsymbol{r}+\boldsymbol{d}_2-\boldsymbol{d}_1|} - \frac{1}{|\boldsymbol{d}_1|} - \frac{1}{|\boldsymbol{d}_2|} - \frac{1}{|\boldsymbol{r}+\boldsymbol{d}_2|} - \frac{1}{|\boldsymbol{d}_1-\boldsymbol{r}|}\right) \tag{S.7}$$

First comes from the interaction of two positive charges, the 2nd from two negative charges, and the rest are all the possible interactions between positive and negative charges (just follow the vectors to know which is which :)).

2. Now prove that for $|\boldsymbol{r}| >> |\boldsymbol{a}|$ we have the following expansion:

$$\frac{1}{|\boldsymbol{r}+\boldsymbol{a}|} = \frac{1}{|\boldsymbol{r}|}\left(1 - \frac{\hat{r}\cdot\boldsymbol{a}}{|\boldsymbol{r}|}\right) + \frac{1}{2|\boldsymbol{r}|^3}\left(3(\hat{r}\cdot\boldsymbol{a})^2 - |\boldsymbol{a}|^2\right) + O\left(\left(\frac{|\boldsymbol{a}|}{|\boldsymbol{r}|}\right)^3\right) \tag{4}$$

Solution. This is rather straightforward. We first need:

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} + O(x^3) \tag{S.8}$$

And then:

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{a}|} = \frac{1}{r\sqrt{1+\frac{2\boldsymbol{a}\cdot\hat{\boldsymbol{r}}}{r}+\frac{\boldsymbol{a}^2}{r^2}}} \tag{S.9}$$

where $\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{|\boldsymbol{r}|}$. Combining those two formulas we get the desired result:

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{a}|} = \frac{1}{|\boldsymbol{r}|}\left(1 - \frac{\boldsymbol{a}\cdot\hat{\boldsymbol{r}}}{|\boldsymbol{r}|} - \frac{\boldsymbol{a}^2}{2|\boldsymbol{r}|^2} + \frac{3(\boldsymbol{a}\cdot\hat{\boldsymbol{r}})^2}{2|\boldsymbol{r}|^2}\right) + ... = \frac{1}{|\boldsymbol{r}|}\left(1 - \frac{\boldsymbol{a}\cdot\hat{\boldsymbol{r}}}{|\boldsymbol{r}|}\right) + \frac{1}{2|\boldsymbol{r}|^3}\left(3(\boldsymbol{a}\cdot\hat{\boldsymbol{r}})^2 - |\boldsymbol{a}|^2\right) + O\left(\left(\frac{|\boldsymbol{a}|}{|\boldsymbol{r}|}\right)^3\right) \tag{S.10}$$

3. Use the result of the previous point to simplify the expression for the energy of the configuration you obtained in the first part of the exercise. To this end, assume that $|\boldsymbol{r}| >> |\boldsymbol{d}_1|,\ |\boldsymbol{d}_2|$.

Solution. We really just blindly use the given formula for the terms $\frac{1}{|\boldsymbol{r}-\boldsymbol{d}_1|}$, $\frac{1}{|\boldsymbol{r}+\boldsymbol{d}_2|}$ and $\frac{1}{|\boldsymbol{r}+\boldsymbol{d}_2-\boldsymbol{d}_1|}$. Most of the terms simplify and we are left with:

$$4\pi\epsilon_0 U = -\frac{q^2}{|\boldsymbol{d}_1|} - \frac{q^2}{|\boldsymbol{d}_2|} - \frac{q^2}{|\boldsymbol{r}|^3}\left(3(\boldsymbol{a}\cdot\hat{\boldsymbol{r}})^2 - \boldsymbol{d}_1\cdot\boldsymbol{d}_2\right) \tag{S.11}$$

4. In the new expression you have just obtained identify the terms that are due to the interactions between two dipoles. We will call their sum the *interaction energy* U_{int}.

Solution. Clearly the first two terms in the energy are due to the inner energy of the respective dipoles, and so the third term is the one responsible for the interaction energy.

5. Now express the scalar products between the vectors $\boldsymbol{r}$, $\boldsymbol{d}_1$, $\boldsymbol{d}_2$ using trigonometric functions in order to obtain for the U_{int} the following expression:

$$U_{int} = \frac{q^2}{4\pi\epsilon_0}\frac{|\boldsymbol{d}_1||\boldsymbol{d}_2|}{|\boldsymbol{r}|^3}(\sin\theta_1\sin\theta_2 - 2\cos\theta_1\cos\theta_2) \tag{5}$$

Solution. Clearly we have $\boldsymbol{d}_{1,2}\cdot\hat{\boldsymbol{r}} = |\boldsymbol{d}_{1,2}|\cos\theta_{1,2}$. Also, the angle between $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ is equal to $\theta_2 - \theta_1$. Therefore $\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = |\boldsymbol{d}_1||\boldsymbol{d}_2|\cos\theta_2 - \theta_1 = |\boldsymbol{d}_1||\boldsymbol{d}_2|(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$. Plug this into the formula given two solutions above to obtain the desired result.

6. The physical systems tend to minimize the interaction energy. Find the values of θ_1 and θ_2 such, that U_{int} is at it's lowest.

Solution. The energy will be at the minimum when $\theta_{1,2} = 0$, which corresponds to all the charges positioned at one axis in the order (+-+-).

7. Finally calculate the force $\boldsymbol{F}$ with which the dipoles act at each other (assume that the angles $\theta_{1,2}$ already satisfy the minimizing condition you found in the previous part) by the means of the formula $\boldsymbol{F} = -\boldsymbol{\nabla}U_{int}$. This is a good approximation as long as the distance between the dipoles is large enough. What further corrections should we include if the dipoles come closer to each other?

Solution. In the minimum the interaction energy is equal to:

$$U_{int} = \frac{-q^2|\boldsymbol{d}_1||\boldsymbol{d}_2|}{2\pi\epsilon_0|\boldsymbol{r}|^3} \tag{S.12}$$

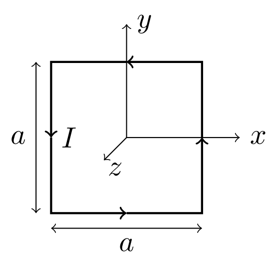
Taking the derivative with respect to r (and an additional minus sign, since $F = -\frac{dU}{dr}$) we obtain:

$$F = -\frac{3q^2\boldsymbol{d}_1||\boldsymbol{d}_2|}{2\pi\epsilon_0 r^4} \tag{S.13}$$

thus getting the binding (well, actually the fact that the dipole is stable was already clear from the U_{int} at minimums). If r decreases too much, we should add higher order terms in the expansions performed earlier on, as well as possibly consider quantum corrections.

Exercise 3. *Magnetic dipole*

Consider a square current loop in the xy-plane of side a. The center of the square is placed in the origin and the sides are parallel to the coordinate-axes.



1. Show that for $r \gg a$ the vector potential $\vec{A}$ is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{\mu} \times \vec{r}}{r^3}, \tag{6}$$

where $\vec{\mu} = a^2 I \vec{e}_z$ and I is the intensity of the current that flows in the loop.
Hint: $\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_\gamma \frac{d\vec{r}'}{|\vec{r}-\vec{r}'|}$.

Solution. The path γ along the loop can be parametrized in the following way:

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, \tag{S.14}$$

where

$$\vec{\gamma}_1(t) = \frac{a}{2}\begin{pmatrix}0\\1\\0\end{pmatrix} - t\begin{pmatrix}1\\0\\0\end{pmatrix} \qquad -\frac{a}{2} \le t \le \frac{a}{2},$$

$$\vec{\gamma}_2(t) = -\frac{a}{2}\begin{pmatrix}1\\0\\0\end{pmatrix} - t\begin{pmatrix}0\\1\\0\end{pmatrix} \qquad -\frac{a}{2} \le t \le \frac{a}{2},$$

$$\vec{\gamma}_3(t) = -\frac{a}{2}\begin{pmatrix}0\\1\\0\end{pmatrix} + t\begin{pmatrix}1\\0\\0\end{pmatrix} \qquad -\frac{a}{2} \le t \le \frac{a}{2},$$

$$\vec{\gamma}_4(t) = \frac{a}{2}\begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}0\\1\\0\end{pmatrix} \qquad -\frac{a}{2} \le t \le \frac{a}{2}.$$

The i-th part of the integral is then

$$\vec{A}_i(\vec{r}) := \frac{\mu_0 I}{4\pi} \int_{\gamma_i} \frac{d\vec{r}'}{|\vec{r}-\vec{r}'|}. \tag{S.15}$$

The first part evaluates to

$$\begin{aligned}
\vec{A}_1(\vec{r}) &= \frac{\mu_0 I \vec{e}_x}{4\pi} \int_{-a/2}^{a/2} \frac{-dt}{\sqrt{(x+t)^2 + (y-\frac{a}{2})^2 + z^2}} \\
&= \frac{\mu_0 I \vec{e}_x}{4\pi} \int_{-a/2}^{a/2} \frac{-dx'}{\sqrt{x^2+y^2+z^2+2xt+t^2-ay+\frac{a^2}{4}}} \\
&\approx \frac{\mu_0 I \vec{e}_x}{4\pi} \int_{-a/2}^{a/2} \frac{-dt}{r}\left(1 - \frac{xt}{r^2} + \frac{ay}{2r^2}\right),
\end{aligned}$$

where in the last step we used $r = \sqrt{x^2+y^2+z^2} \gg a$. Similarly we find

$$\vec{A}_3(\vec{r}) \approx \frac{\mu_0 I \vec{e}_x}{4\pi} \int_{-a/2}^{a/2} \frac{dt}{r}\left(1 + \frac{xt}{r^2} - \frac{ay}{2r^2}\right).$$

Hence

$$\vec{A}_1(\vec{r}) + \vec{A}_3(\vec{r}) = \frac{\mu_0 I \vec{e}_x}{4\pi r^3} \int_{-a/2}^{a/2} dt\,(2xt - ay) = -\frac{\mu_0 I a^2}{4\pi r^3} y\vec{e}_x.$$

On the other hand

$$\vec{A}_2(\vec{r}) + \vec{A}_4(\vec{r}) = \frac{\mu_0 I a^2}{4\pi r^3} x\vec{e}_y.$$

The total vector potential is

$$\vec{A}(\vec{r}) = \frac{\mu_0 I a^2}{4\pi r^3}(-y\vec{e}_x + x\vec{e}_y) = \frac{\mu_0}{4\pi} I a^2 \frac{\vec{e}_z \times \vec{r}}{r^3}$$

2. Compute the magnetic field $\vec{B}$ for $r \gg a$ and compare it to the electric field $\vec{E}$ of an electric dipole.

Solution. Using that $\frac{\vec{r}}{r^3} = -\nabla\frac{1}{r}$ and that $\vec{\mu}$ is constant, we get

$$\begin{aligned}\left(\nabla \times \left(\vec{\mu} \times \nabla\frac{1}{r}\right)\right)_i &= \epsilon_{ijk}\epsilon_{klm}\mu_l\partial_j\partial_m\frac{1}{r}\\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\mu_l\partial_m\partial_j\frac{1}{r}\\ &= \mu_i\partial_m\partial_m\frac{1}{r} - \partial_m\mu_j\partial_j\frac{1}{r}.\end{aligned}$$

In vector notation

$$\begin{aligned}\nabla \times \left(\vec{\mu} \times \nabla\frac{1}{r}\right) &= \vec{\mu}\Delta\frac{1}{r} - \nabla\left(\vec{\mu}\cdot\nabla\frac{1}{r}\right)\\ &= -4\pi\delta(r)\vec{\mu} + \nabla\left(\vec{\mu}\cdot\frac{\vec{r}}{r^3}\right)\\ &= -4\pi\delta(r)\vec{\mu} + \frac{\vec{\mu}}{r^3} - 3\frac{\vec{\mu}\cdot\vec{r}}{r^5}\vec{r}.\end{aligned}$$

Since we approximated the vector potential for $r \gg a$, the magnetic field is valid only in the region far from the origin, hence the δ-function term doesn't add any contribution.

It follows that, the magnetic field is

$$\vec{B}(\vec{r}) = \nabla \times \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left(3\frac{\vec{\mu}\cdot\vec{r}}{r^5}\vec{r} - \frac{\vec{\mu}}{r^3}\right), \tag{S.16}$$

which is similar to the electric field of a dipole

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(3\frac{\vec{p}\cdot\vec{r}}{r^5}\vec{r} - \frac{\vec{p}}{r^3}\right), \tag{S.17}$$

Exercise 4. *The Cosmic Microwave Background (CMB) Sky*

The theoretical and experimental CMB power spectra are customarily presented in the context of spherical harmonic multipoles. In the following link

`http://find.spa.umn.edu/~pryke/logbook/20000922/`

you can find examples of multipoles plots and an application to the CMB Sky. Read it carefully and try to understand the link between harmonic multipoles and one of their application in cosmology.