Name: **Student number:**

Exercise	Max. points	Points	Visum 1	Visum 2
1	15			
2	15			
3	15			
4	15			
Total	**60**			

- The exam lasts 180 minutes.
- Start every new exercise on a new sheet.
- Write your name on every sheet you hand in.
- Do not use red color or pencil.
- You are only allowed to have a single two-sided handwritten formular.

Exercise 1. ***Spherical conductor inside a uniform electric field*** *[15 points]*

The general solution of the Laplace equation in spherical coordinates is

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^l + \frac{B_{lm}}{r^{l+1}}\right)Y_{lm}(\theta,\phi), \tag{1}$$

where the spherical harmonics are given by

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\,\mathrm{e}^{im\phi}. \tag{2}$$

a) (1 point) How does Eq. (1) simplify for potentials with azimuthal symmetry?

A spherical conductor of radius R is placed in an (initially) uniform electric field $\mathbf{E} = E_0\hat{z}$. We place the origin of the coordinate system at the center of the sphere and we denote by θ the polar angle with respect to the $\hat{z}$-axis. The potential of the conductor is set to zero $\Phi(r = R) = 0$.

b) (3 points) What is the potential $\Phi(\mathbf{r})$ very far away from the conductor ($r \gg R$)?

c) (3 points) Show that the potential outside the sphere is given by

$$\Phi(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta + C\left(1 - \frac{R}{r}\right), \tag{3}$$

where C is a constant.

Recall: The Legendre polynomials $P_l(x)$ are given by

$$P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}\left(x^2-1\right)^l. \tag{4}$$

d) (3 points) Find the induced charge density $\sigma(\theta)$ on the surface of the sphere, if the net charge of the conductor is Q. What is the constant C of Eq (3) in that case?

e) (2 points) Calculate from first principles (e.g. Gauss' law) the pressure exerted on a surface element of the sphere for $Q = 0$.

f) (3 points) What is the force acting on the north hemisphere of the sphere and what is the total force acting on the sphere for $Q = 0$?

Solution.

a) In the case of azimuthal symmetry the potential does not depend on ϕ, thus we only have the term $m = 0$ in the above sum. Moreover, since the functions Y_{l0} and P_l only differ by a normalization constant, we introduce new parameters A_l and B_l instead of A_{l0} and B_{l0}. The resulting expression is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta). \tag{S.1}$$

b) At large distances from the sphere the potential does not go to zero, but we must recover the unperturbed uniform field $\mathbf{E} = E_0\hat{z}$. The potential (recall: $\mathbf{E} = -\nabla\Phi$) in this case is thus given by $\Phi(r \gg R) = -E_0 z + C = -E_0 r\cos\theta + C$, where θ is the angle with respect to the $\hat{z}$-axis.

c) The first condition we use is the surface potential $\Phi(r = R) = 0$ and Eq. (S.1). They implies

$$A_l R^l + \frac{B_l}{R^{l+1}} = 0 \quad \Rightarrow \quad B_l = -A_l R^{2l+1}. \tag{S.2}$$

Eq. (S.1) can thus be rewritten as

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta). \tag{S.3}$$

Using now as second condition the one obtained in part (b), and noting that for $r \gg R$ the second term in parentheses in Eq. (S.3) is negligible, we have

$$-E_0 r \cos\theta + C = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta). \tag{S.4}$$

Knowing that $P_0(\cos\theta) = 1$ and $P_1(\cos\theta) = \cos\theta$ we conclude that $A_0 = C$, $A_1 = -E_0$ and $A_k = 0 \;\forall\; k \geq 2$. Finally we get

$$\Phi(r,\theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos\theta + C \left(1 - \frac{R}{r} \right). \tag{S.5}$$

d) Here we use the discontinuity of the electric field across a surface charge σ, which is described by the equation $\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \sigma/\epsilon_0\, \hat{n}$, where *above* and *below* refers to the regions of space separated by the surface and close to it. The vector $\hat{n}$ is a unit vector pointing "upwards". [It is simply derived taking a small enough surface element in which σ is constant and applying Gauss' and Stokes' theorems with a thin pillbox resp. thin rectangular loop.] Using the fact that $\mathbf{E} = -\nabla\Phi$ and that the field inside (here $\mathbf{E}_{\text{below}}$) is zero allows us to find the induced charge density σ by looking at the derivative of the potential in the direction normal to the surface. Using Eq. (S.5) we obtain

$$\sigma(\theta) = -\epsilon_0 \frac{\partial\Phi}{\partial r}\bigg|_{r=R} = 3\epsilon_0 E_0 \cos\theta - \epsilon_0 \frac{C}{R}. \tag{S.6}$$

Now we use the fact that the net charge of the sphere is Q, which gives

$$Q = \int_S \sigma \,\mathrm{d}s = 3\epsilon_0 E_0 \int_S \cos\theta \,\mathrm{d}s - \epsilon_0 \frac{C}{R} \int_S \mathrm{d}s. \tag{S.7}$$

The first integral gives 0, while the second one simply gives $4\pi R^2$. We thus see that $C = -\frac{Q}{4\pi\epsilon_0 R}$ and the potential in Eq. (3) reads

$$\Phi(r,\theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos\theta + \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r} - \frac{1}{R} \right). \tag{S.8}$$

e) We start by considering the force acting on an infinitesimal surface element due to the presence of everything else, except the element itself (since it does not produce a force on itself). The force per unit surface (electrostatic pressure) is given by $\mathbf{f} = \sigma \mathbf{E}_{\text{av}}$, where $\mathbf{E}_{\text{av}} = (\mathbf{E}_{\text{in}} + \mathbf{E}_{\text{out}})/2$ is the average field at the position of the surface element. Here, since we have a conducting sphere, we have that $\mathbf{E}_{\text{in}} = 0$ and the field just above the conducting surface is given by $\mathbf{E}_{\text{out}} = \sigma/\epsilon_0\, \hat{n}$ (again from the discontinuity of the electric field across a surface charge: $\mathbf{E}_{\text{out}} - \mathbf{E}_{\text{in}} = \sigma/\epsilon_0\, \hat{n}$). Therefore the total pressure on a surface element is given by the expression

$$P = \frac{\sigma^2}{2\epsilon_0}, \tag{S.9}$$

f) From symmetry, the components of the final force in the xy-plane do compensate and the resulting force only has a z-component given by

$$f_z = \frac{\sigma^2}{2\epsilon_0} \cos\theta \,\mathrm{d}s = \frac{9}{2} \epsilon_0 E_0^2 \cos^3\theta \,\mathrm{d}s, \tag{S.10}$$

where θ is the angle between $\hat{n}$ and $\hat{z}$ and in the last step we used the surface charge density $\sigma(\theta)$ found in part (d) when Q, i.e. C, is 0. We now integrate Eq. (S.10) over the surface of the northern hemisphere (NH) in order to get the total force

$$F = \int_{\text{NH}} f_z = \frac{9}{2} \epsilon_0 E_0^2 R^2 \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi/2} \cos^3\theta \sin\theta \,\mathrm{d}\theta. \tag{S.11}$$

The integral over θ is easily solved using the substitution $u = \cos\theta$ ($\mathrm{d}u = -\sin\theta\,\mathrm{d}\theta$) giving

$$F = 9\pi\epsilon_0 E_0^2 R^2 \int_0^1 u^3 \mathrm{d}u = \frac{9}{4}\pi\epsilon_0 E_0^2 R^2. \tag{S.12}$$

The total force acting on the sphere is found in the same way, but the integral over θ runs between 0 and π instead of 0 and $\pi/2$, therefore giving $F = 0$.

Exercise 2. ***Magnetic field of a charged rotating spherical shell*** *[15 points]*

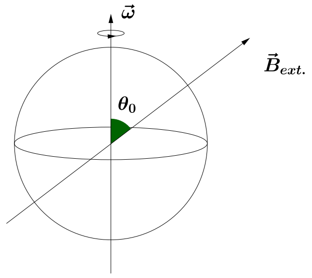

Figure 1: *Charged rotating spherical shell. The external magnetic field refers only to part 5.*

Consider a spherical shell of radius R with a charge Q uniformly distributed on its surface, which rotates around its diameter with constant angular velocity $\vec{\omega}$ (see Fig. 1).

a) (2 points) Calculate the current density $\vec{j}$.

b) (4 points) Show that the vector potential $\vec{A}$ inside and outside the sphere is

$$\vec{A} = \begin{cases} \frac{Q}{12\pi\epsilon_0 c^2 R}\vec{\omega} \times \vec{r} & r < R \\ \\ \frac{QR^2}{12\pi\epsilon_0 c^2}\vec{\omega} \times \frac{\vec{r}}{r^3} & r > R \end{cases}$$

The following integral may be used:

$$\int_{-1}^{1} \frac{\xi d\xi}{(1-\alpha\xi)^{1/2}} = \frac{2}{3\alpha^2}\Big((2-\alpha)\sqrt{1+\alpha} - (2+\alpha)\sqrt{1-\alpha}\Big).$$

c) (4 points) Determine the magnetic field $\vec{B}$ in the interior and the exterior of the spherical shell.

d) (2.5 points) Determine the magnetic dipole moment $\vec{m}$ for the rotating spherical shell.

e) (2.5 points) Assume now that the sphere is immersed in an external magnetic field

$$\vec{B}_{ext.} = \frac{k}{r}\vec{e}_l$$

where k is a constant and the angle between $\vec{e}_l$ and $\vec{\omega}$ is θ_0.Calculate the force acting on the surface of the sphere due to the external magnetic field and determine the associated potential energy.

Solution.

a) The current density is given by $\vec{j} = \rho\vec{v}$, $\vec{v} = \vec{\omega} \times \vec{r}$, and the charge density is $\rho = \sigma\delta(r - R)$ with $\sigma = Q/(4\pi R^2)$. Then we get

$$\vec{j}(\vec{r'}) = \frac{Q}{4\pi R^2}\delta(r' - R)(\vec{\omega} \times \vec{r'}), \tag{S.13}$$

b) For the vector potential we have

$$\vec{A} = \frac{1}{4\pi\epsilon_0 c^2}\int \frac{\vec{j}(\vec{r'})}{|\vec{r}-\vec{r'}|}dV'. \tag{S.14}$$

and so we obtain

$$\vec{A} = \frac{Q}{c^2 16\pi^2\epsilon_0 R^2}\vec{\omega}\times\vec{F}(\vec{r}) \tag{S.15}$$

with

$$\vec{F}(\vec{r}) = \int \frac{\delta(r'-R)}{|\vec{r}-\vec{r'}|}\vec{r'}dV'. \tag{S.16}$$

Because of rotation symmetry it follows that $\vec{F} = f\cdot\vec{r}/r$. We choose the $\vec{r}$-direction to be the z'-axis, then θ' is the angle between $\vec{r}$ and $\vec{r'}$. By using dV' in spherical coordinates and integrating over ϕ' we get

$$F = \int_0^\pi d\theta' \int_0^\infty dr' \frac{\delta(r'-R)r'\cos\theta' 2\pi r'^2\sin\theta'}{(r^2+r'^2-2rr'\cos\theta')^{1/2}} \tag{S.17}$$

$$= \int_0^\pi d\theta' \frac{2\pi R^3\cos\theta'\sin\theta'}{(r^2+R^2-2rR\cos\theta')^{1/2}}. \tag{S.18}$$

We make the substitution $\cos\theta' = \xi$ and we get

$$F = -2\pi R^3\int_{+1}^{-1}\frac{\xi d\xi}{(r^2+R^2-2rR\xi)^{1/2}}. \tag{S.19}$$

This is an integral of the form

$$I = \int\frac{\xi d\xi}{(A-B\xi)^{1/2}} = \frac{1}{\sqrt{A}}\int\frac{\xi d\xi}{(1-\alpha\xi)^{1/2}}, \tag{S.20}$$

with $A = r^2+R^2$, $B = 2r\cdot R$, $\alpha = B/A$. We can then use the hint and finally obtain

$$F = \frac{2\pi R}{3r^2}\Big((r^2+R^2-rR)(r+R) - (r^2+R^2+rR)|r-R|\Big). \tag{S.21}$$

Inside the sphere is $r < R$ and so $|r-R| = R-r$ and

$$F = \frac{4\pi}{3}Rr. \tag{S.22}$$

Outside we find

$$F = \frac{4\pi}{3}\frac{R^4}{r^2}. \tag{S.23}$$

The solution for the vector potential is

$$\vec{A} = \begin{cases} \frac{Q}{12\pi\epsilon_0 c^2 R}\vec{\omega}\times\vec{r} & r < R \\ \frac{QR^2}{12\pi\epsilon_0 c^2}\vec{\omega}\times\frac{\vec{r}}{r^3} & r > R \end{cases}$$

c) We can then get the magnetic field $\vec{B}$ by taking $\vec{B} = \vec{\nabla}\times\vec{A}$,

$$\vec{B} = \frac{Q}{12\pi\epsilon_0 c^2 R}\Big(\vec{\omega}(\vec{\nabla}\cdot\vec{r}) - (\vec{\omega}\cdot\vec{\nabla})\vec{r}\Big) \qquad r < R. \tag{S.24}$$

Which gives, after explicit computation,

$$\vec{B} = \frac{Q}{6\pi\epsilon_0 c^2 R}\vec{\omega} \qquad r < R. \tag{S.25}$$

For $r > R$ we obtain,

$$\vec{B} = \frac{QR^2}{12\pi\epsilon_0 c^2}\frac{3\Big(\vec{\omega}\cdot\frac{\vec{r}}{r}\Big)\frac{\vec{r}}{r} - \vec{\omega}}{r^3} \equiv \frac{3(\vec{m}\cdot\vec{r})\vec{r} - r^2\vec{m}}{r^5}. \tag{S.26}$$

d) We see that the $\vec{B}$-field inside the sphere is homogeneous and proportional to ω. Outside the sphere we find the behaviour of a dipole with

$$\vec{m} = \frac{QR^2\vec{\omega}}{12\pi\epsilon_0 c^2}. \tag{S.27}$$

e) The potential is the one acting on the dipole moment and is given by

$$U = -\vec{m}\cdot\vec{B} = -\frac{QR^2}{12\pi\epsilon_0 c^2}\vec{\omega}\cdot\vec{B}_{ext.} = -\frac{QR^2}{12\pi\epsilon_0 c^2}\omega B_{ext.}\cos(\theta_0) = -\frac{QRk}{12\pi\epsilon_0 c^2}\omega\cos(\theta_0). \tag{S.28}$$

The corresponding force is

$$\vec{F} = -\nabla U = -\frac{kQ}{12\pi\epsilon_0 c^2}\omega\cos(\theta_0)\vec{e}_r \tag{S.29}$$

Exercise 3. ***Special Relativity*** *[15 points]*

A point-like charge q of mass m is initially at rest and it gets accelerated by a constant force $\vec{F}$.

a) (2 points) What is the velocity of the charge as a function of time?

b) (2 points) What is the energy of the charge as a function of time?

c) (2 points) What is the distance travelled by the charge as a function of its energy?

d) (2 points) Calculate the derivative
$$\frac{d\tau}{dE}, \tag{5}$$
where τ is the proper time of the particle and E the energy of the charge, as a function of the energy.

e) (4 points) Calculate the instantaneous force felt by the charge in its own rest frame during the linear acceleration.

f) (3 points) Assume now that the constant force $\vec{F}$ originates from a constant electric field. What is the electromagnetic field strength tensor $F^{\mu\nu}$ in the instantaneous rest frame of the charge? Is your answer consistent with the result of the previous question?

Hint. Recall that
$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix}.$$

Solution.

a) Since $F = dp/dt$, we have that $p(t) = Ft$. Now on the other hand $p(t) = \gamma(t)mv(t) = \frac{mv}{\sqrt{1-v^2/c^2}}$, and so solving for v yields $v(t) = \frac{Ft}{m\sqrt{1+F^2t^2/(m^2c^2)}}$.

b) The energy is given by
$$E(t) = \gamma mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} = \frac{mc^2}{\sqrt{1-\frac{F^2t^2}{m^2c^2+F^2t^2}}} = \frac{mc^2\sqrt{m^2c^2+F^2t^2}}{mc} = mc^2\sqrt{1+\frac{F^2t^2}{m^2c^2}}.$$

c) To find the distance, integrate $v(t')$ up to time t. We get
$$x(t) = \int_0^t v(t')dt' = \left[\frac{mc^2}{F}\sqrt{1+\frac{F^2t^2}{m^2c^2}}\right]_0^t.$$
In terms of the energy, this yields
$$x(E) = \frac{E-mc^2}{F} = \frac{E_{\text{kin}}}{F}.$$
This result could have also been obtained through $E = \int F dx$ together with the knowledge that F is constant.

d) To get the proper time as a function of the energy, note that $d\tau = \frac{dt}{\gamma}$ and so $\frac{d\tau}{dE} = \frac{1}{\gamma}\frac{dt}{dE}$. Then, we use
$$dE/dt = \frac{F^2t}{m\sqrt{1+\frac{F^2t^2}{m^2c^2}}} = \frac{Fp}{m\sqrt{1+\frac{F^2t^2}{m^2c^2}}}$$
to get
$$\frac{d\tau}{dE} = \frac{m}{\gamma}\frac{\sqrt{1+\frac{F^2t^2}{m^2c^2}}}{Fp},$$

yielding

$$\frac{d\tau}{dE} = \frac{E}{c^2\gamma F p} = \frac{E}{Fc^2\gamma\sqrt{\gamma^2-1}mc} = \frac{1}{Fc\sqrt{E^2/(mc^2)-1}} = \frac{m}{F\sqrt{(E^2-m^2c^4)}}$$

as $p = \gamma m v = \sqrt{\gamma^2-1}mc$.

e) We calculate this using the four-force, but we can do this both starting from the rest frame of the particle or from the lab frame. In the rest frame, the four-force takes the simple form $f'^\mu = (0, \vec{F}')$. Assuming the particle moves in the x direction, we can transform this with a Lorentz transformation of speed v to the lab frame and obtain $f^\mu = L^\mu_\nu f'^\nu = (\gamma v F'_x, \gamma F'_x, F'_y, F'_z)$. Now we know that in the lab frame $f^i = \gamma F_i$ for $i \in \{1,2,3\}$, and so $F_x = F'_x$, $F'_y = \gamma F_y = 0$ and $F'_z = \gamma F_z = 0$.

To calculate it starting from the lab frame is slightly more complicated: the four-force in the laboratory frame is given by $f^\mu = (mc\frac{d\gamma}{d\tau}, \gamma\vec{F})$. For convenience, we shall denote by $\vec{\gamma}$ the vector $(\gamma, 0, 0)$, and we shall look at the force on the particle when it is travelling in x-direction (without loss of generality). Now, as we change frame into the particle rest frame, we use a Lorentz transformation of speed v in x-direction, yielding

$$f'^\mu = L^\mu{}_\nu f^\nu = (mc\gamma\frac{d\gamma}{d\tau} - \gamma\beta\vec{\gamma}\cdot\vec{F}, -mc\gamma\beta\frac{d\gamma}{d\tau} + \gamma^2 F_x, \gamma F_y, \gamma F_z).$$

In the case of linear acceleration, $\vec{F} = (F, 0, 0)$, $\vec{\gamma}\cdot\vec{F} = \gamma F$ and

$$\frac{d\gamma}{d\tau} = \gamma\frac{d\gamma}{dt} = \frac{F^2 t\gamma}{mc}(m^2c^2 + F^2t^2)^{-1/2} = \frac{F^2 t}{m^2c^2},$$

yielding an overall force four vector of

$$f'^\mu = (mc\gamma\frac{F^2t}{m^2c^2} - \gamma^2\beta F, -mc\gamma\beta\frac{F^2t}{m^2c^2} + \gamma^2 F, 0, 0).$$

The force felt by the particle is hence

$$F' = -mc\gamma\beta\frac{F^2t}{m^2c^2} + \gamma^2 F = \gamma^2 F - \frac{F^3t^2}{m^2c^2} = F$$

in x-direction.

f) The electric field in this case transforms as $E'_\parallel = E_\parallel$ for the linear motion, so the results indeed coincide. To calculate the result with the more general transformation rule, use $F'^{\mu\nu} = L^\mu{}_\rho L^\nu{}_\sigma F^{\rho\sigma}$:

For our case of linear acceleration in an E field, the B field vanishes and $E^1 = E$, so the tensor F takes the simple form

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

With

$$L^\mu{}_\nu = \begin{bmatrix} \gamma & -p/m & 0 & 0 \\ -p/m & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we get

$$F'^{\mu\nu} = L^{\mu}{}_{\rho} L^{\nu}{}_{\sigma} F^{\rho\sigma} = \begin{bmatrix} \gamma & -p/m & 0 & 0 \\ -p/m & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & -p/m & 0 & 0 \\ -p/m & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & -p/m & 0 & 0 \\ -p/m & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Ep/m & -E\gamma & 0 & 0 \\ E\gamma & -Ep/m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -E\gamma^2 + Ep^2/m^2 & 0 & 0 \\ E\gamma^2 - Ep^2/m^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which yields $E'_{\parallel} = E_{\parallel}$.

We could have alternatively employed tensor notation to solve this question. Then, we would only have to check the relevant components though (and not need to do the whole transformation).

Exercise 4. *Radiation emitted by an accelerated charge* *[15 points]*

a) (2 points) The electric $\vec{E}$ field generated by a moving charge q is given by

$$\vec{E}(\vec{x}, t) = \frac{q}{4\pi} \left[\frac{\vec{n} \times \{(\vec{n} - \vec{v}) \times \dot{\vec{v}}\}}{(1 - \vec{v} \cdot \vec{n})^3 R} \right]_{ret} + O(1/R^2) \tag{6}$$

where v is the velocity of the charge,

$$\gamma = (1 - v^2)^{-1/2} \tag{7}$$

$$\vec{R} = \vec{n} R, \quad \text{with} \quad R = |\vec{x} - \vec{r}(t_{ret})|, \tag{8}$$

and $\vec{r}$ is the position vector of the charge. All the quantities on the right hand side of Eq. 6 are computed at a "retarded time" t_{ret} while the electric field on the left hand side is evaluated at an observation time t.

- What is the equation relating the retarded time and the observation time?
- Prove that

$$\frac{dt}{dt_{ret}} = 1 - \vec{v} \cdot \vec{n}.$$

b) (2 points) Show that the Poynting vector for a moving charge can be written as

$$\vec{S} = \left| \vec{E} \right|^2 \vec{n} - \left(\vec{E} \cdot \vec{n} \right) \vec{E}. \tag{9}$$

c) (2 points) Starting from the continuity equation satisfied by the Poynting vector show that the energy radiated per unit of retarded time and unit of solid angle centered around the moving charge at the retarded time is given by

$$\frac{dP_{ret}}{d\Omega} \equiv \frac{dE}{dt_{ret} d\Omega} = \left[\vec{S} \cdot \vec{n} R^2 \right]_{ret} \frac{dt}{dt_{ret}}. \tag{10}$$

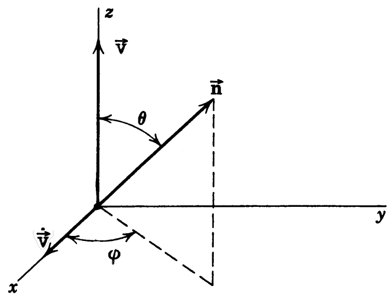

Figure 2: Angles between $\dot{\vec{v}}, \vec{v}, \vec{n}$

d) (4 points) Assume that the velocity of the charge is perpendicular to the acceleration, as in Fig. 2. Show that the power radiated by the charge as a function of the angles θ, φ, shown in Fig. 2, is

$$\frac{dP_{ret}}{d\Omega} = \frac{q^2}{16\pi^2} \frac{|\dot{\vec{v}}|^2}{(1-v\,\cos\theta)^3}\left[1 - \frac{\sin^2\theta\,\cos^2\varphi}{\gamma^2\,(1-v\,\cos\theta)^2}\right]. \tag{11}$$

Tip: It may be helpful to prove first the identity

$$\left|\vec{n}\times(\vec{c}\times\vec{d})\right|^2 = \vec{c}^2(\vec{n}\cdot\vec{d})^2 + \vec{d}^2(\vec{n}\cdot\vec{c})^2 - 2(\vec{c}\cdot\vec{d})(\vec{n}\cdot\vec{c})(\vec{n}\cdot\vec{d}) \tag{12}$$

e) (2 points) Explain qualitatively in which direction the power radiated is stronger for a charge moving with a velocity close to the speed of light.

f) (3 points) What is the total power radiated in all directions?

Solution.

a)
- The equation relating the retarded time and the observation time is

$$t - t_{ret} = R(t_{ret}) \tag{S.30}$$

  with $R(t_{ret})$ defined as in eq. (8).
- We have

$$\begin{aligned} t &= t_{ret} + R(t_{ret}) &\text{(S.31)}\\ \frac{dt}{dt'} &= 1 + \frac{d}{dt_{ret}} R(t_{ret}) \\ &= 1 + \frac{d}{dt_{ret}}\sqrt{(\vec{x}-\vec{r}(t_{ret}))^2} \\ &= 1 + \frac{\vec{x}-\vec{r}(t_{ret})}{|\vec{x}-\vec{r}(t_{ret})|}\cdot(-\vec{v}) \\ &= 1 - \vec{n}\cdot\vec{v} &\text{(S.32)} \end{aligned}$$

b) The Poynting's vector is given by

$$\vec{S} = \vec{E}\times\vec{B} \tag{S.33}$$

where the magnetic field $\vec{B}$ is related to $\vec{E}$ via

$$\vec{B}(\vec{x},t) = [\vec{n}\times\vec{E}]_{ret} \tag{S.34}$$

We then have

$$\vec{S} = \vec{E}\times(\vec{n}\times\vec{E}) \tag{S.35}$$

and in components

$$
\begin{aligned}
S_i &= \epsilon_{ijk}\, E_j (\vec{n} \times \vec{E})_k \\
&= \epsilon_{ijk}\, E_j\, \epsilon_{lmk}\, n_l\, E_m \\
&= (\delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl})\, n_l\, E_j\, E_m
\end{aligned}
$$

that is

$$
\vec{S} = (\vec{n}\,|\vec{E}|^2 - \vec{E}\,(\vec{n} \cdot \vec{E})) \tag{S.36}
$$

c) From the continuity equation for the Poynting vector

$$
\frac{d\omega}{dt} + \vec{\nabla} \cdot \vec{S} = 0 \tag{S.37}
$$

where ω is the energy density, we get

$$
\begin{aligned}
&\int_V \frac{d\omega}{dt} = \frac{dE}{dt} = \int_{\partial V} \vec{S} \cdot \vec{n}\, R(t_{ret}(t))^2\, d\Omega \\
&\frac{dE}{d\Omega\, dt} = \frac{dP}{d\Omega} = \vec{S} \cdot \vec{n}\, R(t_{ret}(t))^2 \\
&\frac{dE}{d\Omega\, dt_{ret}} = \frac{dP}{d\Omega}\, \frac{dt}{dt_{ret}} = \frac{dP_{ret}}{d\Omega} = \vec{S} \cdot \vec{n}\, R(t_{ret})^2\, \frac{dt}{dt_{ret}}
\end{aligned} \tag{S.38}
$$

d) From eqs. (6) and (S.36) the Poynting vector takes here the form

$$
\begin{aligned}
\vec{S} &= (\vec{n}\,|\vec{E}|^2 - \vec{E}\,(\vec{n} \cdot \vec{E})) \\
&= \frac{q^2}{16\pi^2}\, \vec{n} \left[\frac{1}{R^2} \left| \frac{\vec{n} \times \{(\vec{n} - \vec{v}) \times \dot{\vec{v}}\}}{(1 - \vec{v} \cdot \vec{n})^3} \right|^2 \right]_{ret} + O(1/R^3)
\end{aligned} \tag{S.39}
$$

Then the power radiated becomes

$$
\begin{aligned}
\frac{dP_{ret}}{d\Omega} &= (\vec{S} \cdot \vec{n})\, R^2\, \frac{dt}{dt_{ret}} \\
&= (\vec{S} \cdot \vec{n})\, R^2\, (1 - \vec{v} \cdot \vec{n}) \\
&= \frac{q^2}{16\pi^2}\, \frac{\left| \vec{n} \times \left((\vec{n} - \vec{v}) \times \dot{\vec{v}} \right) \right|^2}{(1 - \vec{v} \cdot \vec{n})^5}
\end{aligned} \tag{S.40}
$$

$$
\tag{S.41}
$$

First we have to compute

$$
\begin{aligned}
\left| \vec{n} \times \left((\vec{n} - \vec{v}) \times \dot{\vec{v}} \right) \right|^2 &= (\epsilon_{ijk}\, n_j\, (\epsilon_{klm}\, (n - v)_l \dot{v}_m)) \cdot (\epsilon_{iab}\, n_a\, (\epsilon_{bcd}\, (n - v)_c \dot{v}_d)) \\
&= (\delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl})\, (\delta_{ic}\, \delta_{ad} - \delta_{id}\, \delta_{ac})\, (n - v)_l\, \dot{v}_m\, (n - v)_c\, \dot{v}_d\, n_j\, n_a \\
&= (\delta_{cl}\, \delta_{jm}\, \delta_{ad} - \delta_{dl}\, \delta_{jm}\, \delta_{ac} - \delta_{cm}\, \delta_{jl}\, \delta_{ad} + \delta_{dm}\, \delta_{jl}\, \delta_{ac})\, (n - v)_l\, \dot{v}_m\, (n - v)_c\, \dot{v}_d\, n_j\, n_a \\
&= (\vec{n} - \vec{v})^2\, (\vec{n} \cdot \dot{\vec{v}})^2 - 2 (\vec{n} \cdot \dot{\vec{v}}) \left(\dot{\vec{v}} \cdot (\vec{n} - \vec{v}) \right) (\vec{n} \cdot (\vec{n} - \vec{v})) + (\dot{\vec{v}})^2\, (\vec{n} \cdot (\vec{n} - \vec{v}))^2
\end{aligned} \tag{S.42}
$$

Now using the fact that $\dot{\vec{v}} \perp \vec{v}$ and $|\vec{n}|^2 = 1$, the previous expression reduces to

$$
\left| \vec{n} \times \left((\vec{n} - \vec{v}) \times \dot{\vec{v}} \right) \right|^2 = (\vec{n} - \vec{v})^2\, (\vec{n} \cdot \dot{\vec{v}})^2 - 2 (\vec{n} \cdot \dot{\vec{v}})^2\, (1 - \vec{n} \cdot \vec{v}) + (\dot{\vec{v}})^2\, (1 - \vec{n} \cdot \vec{v})^2 \tag{S.43}
$$

Using the polar angles as in the picture, we have

$$
\vec{n} \cdot \vec{v} = v\, \cos\theta \tag{S.44}
$$

$$
\vec{n} \cdot \dot{\vec{v}} = \dot{v}\, \cos\varphi\, \cos(\pi/2 - \theta) = \dot{v}\, \cos\varphi\, \sin\theta \tag{S.45}
$$

$$
(\vec{n} - \vec{v})^2 = 1 + v^2 - 2\, v\, \cos\theta \tag{S.46}
$$

Inserting these quantities in (S.43), we find

$$
\left| \vec{n} \times \left((\vec{n} - \vec{v}) \times \dot{\vec{v}} \right) \right|^2 = (\dot{\vec{v}})^2\, (1 - v\, \cos\theta)^2 - (\dot{\vec{v}})^2\, \gamma^{-2}\, \sin^2\theta\, \cos^2\varphi \tag{S.47}
$$

with $\gamma = (1 - v^2)^{-1/2}$. Hence the power radiated becomes

$$
\frac{dP_{ret}}{d\Omega} = \frac{q^2}{16\pi^2}\, \frac{|\dot{\vec{v}}|^2}{(1 - v\, \cos\theta)^3} \left[1 - \frac{\sin^2\theta\, \cos^2\varphi}{\gamma^2\, (1 - v\, \cos\theta)^2} \right]. \tag{S.48}
$$

e) In the relativistic limit ($v \to c$) the maximal power radiated is obtained in the forward direction ($\theta \to 0$), as in the case of linear acceleration.

f) The integral we have to compute is

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta \left(\frac{q^2}{16\pi^2} \frac{|\dot{\vec{v}}|^2}{(1 - v\, \cos\theta)^3} \left[1 - \frac{\sin^2\theta\, \cos^2\varphi}{\gamma^2\, (1 - v\, \cos\theta)^2} \right] \right) \tag{S.49}$$

The first term in the brackets will give

$$\begin{aligned} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \frac{\sin\theta}{(1 - v\, \cos\theta)^3} &= 2\pi \left(-\frac{1}{2v} (1 - v\, \cos\theta)^{-2} \right) \Big|_0^{\pi} \\ &= 2\pi \left(-\frac{1}{2v} \left[\frac{1}{(1+v)^2} - \frac{1}{(1-v)^2} \right] \right) \\ &= 2\pi \left(-\frac{1}{2v} \frac{-4v}{(1-v^2)^2} \right) \\ &= \frac{4\pi}{(1-v^2)^2} \end{aligned} \tag{S.50}$$

The second term gives

$$\begin{aligned} &\int_0^{2\pi} d\varphi \, \cos^2\varphi \int_0^{\pi} d\theta \, \sin\theta \frac{\sin^2\theta}{(1 - v\, \cos\theta)^5} = \\ &= \int_0^{2\pi} d\varphi \, \frac{1}{2} (1 + \cos 2\varphi) \int_0^{\pi} d\theta \, \sin\theta \frac{\sin^2\theta}{(1 - v\, \cos\theta)^5} \\ &= \pi \left[\frac{1}{6v^2} \frac{2 + 6v^2}{(1-v^2)^3} - \frac{1}{3v^2} \frac{1}{(1-v^2)^2} \right] \\ &= \pi \, \frac{4}{3} \frac{1}{(1-v^2)^3} \end{aligned} \tag{S.51}$$

Combining the two results:

$$\begin{aligned} P_{ret} &= \frac{q^2}{16\pi^2} |\dot{\vec{v}}|^2 \left(4\pi\gamma^4 - \pi \, \frac{4}{3} \gamma^4 \right) \\ &= \frac{q^2}{4\pi} \frac{2}{3} |\dot{\vec{v}}|^2 \, \gamma^4 \end{aligned} \tag{S.52}$$

that is exactly Larmor's formula for a charge in circular motion.