Exercise 1. *Green's functions for the wave equation*

a) Prove, without using Fourier transforms, the identity

$$\int_{-\infty}^{\infty} d\omega\, e^{-i\omega x} = 2\pi\delta(x). \tag{1}$$

Solution. We will use the fact that, in the sense of distributions,

$$\frac{d}{dx}\Theta(x>0) = \delta(x),$$

where Θ is the Heaviside function. We define the Heaviside function in the following way:

$$\Theta(x>0) = \lim_{\epsilon\to 0} \frac{-1}{2\pi i}\int_{-\infty}^{\infty} d\omega\, \frac{e^{-i\omega x}}{\omega + i\epsilon}. \tag{S.1}$$

Indeed, for $x<0$, the integrand falls off for $\mathrm{Im}(\omega)\to +\infty$, and the pole below the real axis is not contained in the contour integral obtained by closing the loop in the upper half complex plane. By virtue of Cauchy's theorem, the expression (S.1) is thus zero.
For $x>0$, we have to close the contour in the lower half plane, thus picking up the residue of the pole at $-i\epsilon$. The residue is simply 1, and we get an additional minus sign from circling the contour clock-wise. Thus,

$$\frac{-1}{2\pi i}\int_{-\infty}^{\infty} d\omega\, \frac{e^{-i\omega x}}{\omega + i\epsilon} = \frac{2\pi i}{2\pi i} = 1, \tag{S.2}$$

for $x>0$. So we have proven that the above integral defines a representation of the Heaviside function. Now,

$$\begin{aligned}
\delta(x) &= \Theta'(x>0) && \text{(S.3)}\\
&= \lim_{\epsilon\to 0}\frac{-1}{2\pi i}\int_{-\infty}^{\infty} d\omega\, \frac{d}{dx}\frac{e^{-i\omega x}}{\omega+i\epsilon} && \text{(S.4)}\\
&= \lim_{\epsilon\to 0}\frac{-1}{2\pi i}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega x}\frac{-i\omega}{\omega+i\epsilon} && \text{(S.5)}\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega x}. && \text{(S.6)}
\end{aligned}$$

The Green's function for the d' Alembert operator can be written in the following form

$$G(\mathbf{x},t) = \lim_{\delta_1,\delta_2\to 0} c\int \frac{d^3k dE}{(2\pi)^4}\, \frac{-e^{-i(cEt-\mathbf{k}\cdot\mathbf{x})}}{2|\mathbf{k}|}\left[\frac{1}{E-|\mathbf{k}|+i\delta_1} - \frac{1}{E+|\mathbf{k}|+i\delta_2}\right]. \tag{2}$$

The result depends on the choice of sign for the regulators δ_1 and δ_2.

For $\delta_1>0$ and $\delta_2>0$ the result is the *retarded* Green's function

$$G_{ret}(\mathbf{x},t) = \frac{1}{4\pi|\mathbf{x}|}\Theta(t>0)\,\delta\left(t-\frac{|\mathbf{x}|}{c}\right). \tag{3}$$

b) Show that for $\delta_1 < 0$ and $\delta_2 < 0$ the Green's function is the *advanced* Green's function

$$G_{adv}(\mathbf{x}, t) = \frac{1}{4\pi|\mathbf{x}|}\Theta(t<0)\,\delta\left(t + \frac{|\mathbf{x}|}{c}\right). \tag{4}$$

Solution. We first integrate out the angular part of the d^3k integration (aligning the coordinates with $\mathbf{x}$):

$$\begin{aligned}
\int d^3k \frac{e^{-i(cEt - \mathbf{k}\cdot\mathbf{x})}}{2|\mathbf{k}|} &= \int dk d\phi d\theta\, k^2 \sin\theta \frac{e^{-i(cEt - kx\cos\theta)}}{2k} &\text{(S.7)}\\
&= 2\pi e^{-icEt} \int dk\, \frac{k}{2} \int_{-1}^{1} d\cos\theta\, e^{ikx\cos\theta} &\text{(S.8)}\\
&= \frac{-2\pi i}{2x} \int dk\, e^{-icEt}\left(e^{ikx} - e^{-ikx}\right), &\text{(S.9)}
\end{aligned}$$

where $x := |\mathbf{x}|$ and $k := |\mathbf{k}|$. Thus, suppressing the limit of $\delta_{1,2} \to 0$,

$$\frac{G(\mathbf{x}, t)}{c} = \frac{i\pi}{|\mathbf{x}|(2\pi)^4} \int dk dE e^{-icEt}\left(e^{ikx} - e^{-ikx}\right)\left[\frac{1}{E - k + i\delta_1} - \frac{1}{E + k + i\delta_2}\right]. \tag{S.10}$$

Let us consider the case $\delta_1 < 0$, $\delta_2 < 0$, i.e. the pole at $E = +k$ and the pole at $E = -k$ are moved above the real axis. For $t > 0$, we choose the contour in the lower half of the complex plane (otherwise the exponential $\exp(-icEt)$ blows up). Thus, the integral is zero, since no pole are contained inside the contour, i.e.

$$\frac{G_{adv}(\mathbf{x}, t > 0)}{c} = 0. \tag{S.11}$$

For $t < 0$, we choose the contour in the upper half of the complex plane. Thus, we pick up the residue of the two poles, which are equal to 1. We find

$$\begin{aligned}
\frac{G_{adv}(\mathbf{x}, t < 0)}{c} &= \frac{2(i\pi)^2}{|\mathbf{x}|(2\pi)^4} \int_0^\infty dk \left(e^{-ickt}\left(e^{ikx} - e^{-ikx}\right) - e^{ikct}\left(e^{ikx} - e^{-ikx}\right)\right) &\text{(S.12)}\\
&= -\frac{1}{8\pi^2|\mathbf{x}|}\left(\int_0^\infty dk\, e^{-ickt}\left(e^{ikx} - e^{-ikx}\right) - \int_{-\infty}^{0} dk\, e^{-ikct}\left(e^{-ikx} - e^{ikx}\right)\right) &\text{(S.13)}\\
&= -\frac{1}{8\pi^2|\mathbf{x}|}\int_{-\infty}^{\infty} dk \left(e^{-ik(x-ct)} - e^{-ik(x+ct)}\right) &\text{(S.14)}\\
&= -\frac{1}{4\pi|\mathbf{x}|}\left(\delta(x - ct) - \delta(x + ct)\right) &\text{(S.15)}\\
&\overset{t\leq 0}{=} \frac{1}{4\pi|\mathbf{x}|}\delta(x + ct). &\text{(S.16)}
\end{aligned}$$

Hence

$$G_{adv}(\mathbf{x}, t) = \frac{1}{4\pi|\mathbf{x}|}\Theta(t<0)\,\delta\left(t + \frac{|\mathbf{x}|}{c}\right). \tag{S.17}$$

c) Calculate the Green's function for the remaining two cases ($\delta_1 > 0$ and $\delta_2 < 0$ and vice versa). You should find that only the sum of the two can be brought to a similar form as the cases above.

Solution. Let us first consider the case $\delta_1 > 0$, $\delta_2 < 0$ of the integral (S.10), i.e. the pole at $E = +k$ is moved below and the pole at $E = -k$ is moved above the real axis. For $t > 0$, we choose the contour in the lower half of the complex plane (otherwise the exponential $\exp(-icEt)$ blows up). Thus, we pick up the residue of the pole at $E = +k$ which equals 1, whereas the second term in the square brackets does not contribute. With an additional minus sign from the clockwise contour, we find:

$$\frac{G(\mathbf{x}, t > 0)}{c} = -\frac{2(i\pi)^2}{|\mathbf{x}|(2\pi)^4} \int_0^\infty dk e^{-ickt} \left(e^{ikx} - e^{-ikx}\right) = \frac{1}{8\pi^2|\mathbf{x}|} \int_0^\infty \left(e^{ik(x+ct)} - e^{-ik(x-ct)}\right). \quad \text{(S.18)}$$

Unfortunately, we can not extend the integral to the whole range $(-\infty, \infty)$ as in the lecture. Only when we add the contribution from the opposite choice of signs, we can arrive at a representation of the delta function. We therefore define

$$G_0 := G_{\delta_1>0,\delta_2<0} + G_{\delta_1<0,\delta_2>0}, \quad \text{(S.19)}$$

and find for $t > 0$:

$$\begin{aligned}
\frac{G_0(\mathbf{x}, t > 0)}{c} &= \frac{1}{8\pi^2|\mathbf{x}|} \int_0^\infty dk \left(e^{-ickt} - e^{ikct}\right) \left(e^{ikx} - e^{-ikx}\right) && \text{(S.20)} \\
&= \frac{1}{8\pi^2|\mathbf{x}|} \int_{-\infty}^\infty dk \left(e^{-ik(x-ct)} - e^{-ik(x+ct)}\right) && \text{(S.21)} \\
&= \frac{1}{4\pi|\mathbf{x}|} (\delta(x - ct) - \delta(x + ct)) && \text{(S.22)} \\
&\overset{t\geq 0}{=} \frac{1}{4\pi|\mathbf{x}|} \delta(x - ct) = \frac{G_{ret}(\mathbf{x}, t > 0)}{c}. && \text{(S.23)}
\end{aligned}$$

Similarly for $t < 0$

$$\frac{G_0(\mathbf{x}, t < 0)}{c} = \frac{1}{4\pi|\mathbf{x}|} \delta(x + ct) = \frac{G_{adv}(\mathbf{x}, t < 0)}{c}, \quad \text{(S.24)}$$

i.e. $\boxed{G_0 = G_{ret} + G_{adv}}$.

Exercise 2. *Electric Dipole Radiation*

Imagine two tiny metal spheres at distance d from each other connected by a wire (see Figure 1), where at time t, the upper sphere carries a charge $q(t) = q_0 \cos(\omega t)$ while the charge on the lower sphere is given by $-q(t)$.

a) Calculate the electric potential far away from the dipole. Use $d \ll r$ and $d \ll \frac{c}{\omega}$.

b) Take the limit of $\omega \to 0$. What do you expect?

c) Now look at the case where also $r \gg \frac{c}{\omega}$, that is, when we are interested in large distances from the source in comparison to the wavelength ($r \gg \lambda$). How does the expression for the potential simplify in this case?

d) Obtain an expression for the vector potential in the limit $d \ll r$ and $d \ll \frac{c}{\omega}$.

e) Calculate the resulting electric and magnetic fields in the same limit with also $r \gg \frac{c}{\omega}$.

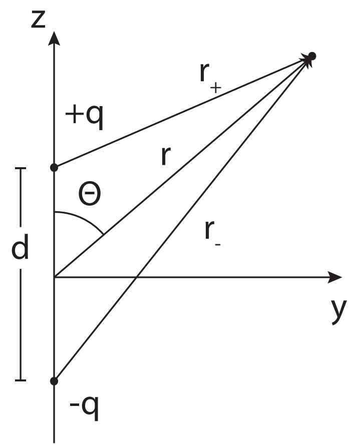

Figure 1: Electric Dipole

Solution.

a) Start from the equation for the scalar potential

$$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}',t_{\tilde{r}})}{\tilde{r}}d^3x. \tag{S.25}$$

The charge density is given by

$$\rho(x,t) = q(t)(\delta(x-\frac{d}{2}e_z) - \delta(x+\frac{d}{2}e_z)) \tag{S.26}$$

and so with $t_{r_\pm} = t - \frac{r_\pm}{c}$ the retarded potential from Equation S.25 becomes

$$\Phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_0\cos(\omega(t-r_+/c))}{r_+} - \frac{q_0\cos(\omega(t-r_-/c))}{r_-}\right\} \tag{S.27}$$

with $r_\pm = \sqrt{r^2 \mp rd\cos\Theta + (d/2)^2}$.

Since $d \ll r$, we have that

$$r_\pm \approx r\left(1 \mp \frac{d}{2r}\cos\Theta\right), \tag{S.28}$$

$$\frac{1}{r_\pm} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\Theta\right), \tag{S.29}$$

as well as

$$\cos(\omega(t-r_\pm/c)) \approx \cos\left(\omega(t-r/c) \pm \frac{\omega d}{2c}\cos\Theta\right) \tag{S.30}$$

$$= \cos(\omega(t-r/c))\cos\left(\frac{\omega d}{2c}\cos\Theta\right) \mp \sin(\omega(t-r/c))\sin\left(\frac{\omega d}{2c}\cos\Theta\right). \tag{S.31}$$

Using $d \ll \frac{c}{\omega}$, this simplifies to

$$\cos(\omega(t-r_\pm/c)) \approx \cos(\omega(t-r/c)) \mp \frac{\omega d}{2c}\cos\Theta\sin(\omega(t-r/c)) \tag{S.32}$$

and so

$$\Phi(r,\Theta,t) \approx \frac{p_0\cos\Theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega}{c}\sin(\omega(t-r/c)) + \frac{1}{r}\cos(\omega(t-r/c))\right\}. \tag{S.33}$$

b) In the limit $\omega \to 0$ we recover the static dipole result

$$\Phi(r,\Theta) = \frac{p_0\cos\Theta}{4\pi\epsilon_0 r^2} \tag{S.34}$$

as expected.

c) Far away from the dipole, $r \gg \frac{c}{\omega}$ implies

$$\Phi(r,\Theta,t) \approx -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\Theta}{r}\right)\sin(\omega(t-r/c)). \tag{S.35}$$

d) We will use that

$$\vec{A}(\vec{x},t) = \frac{1}{4\pi\epsilon_0 c^2}\int \frac{\vec{J}(\vec{x}',t_{\tilde{r}})}{\tilde{r}}d^3x'. \tag{S.36}$$

Now, first of all, we have to compute the current flowing into the wire:

$$\vec{I}(\vec{z}) = \frac{dq}{dt}\hat{z} = -q_0\,\omega\,\sin(\omega t)\,\hat{z} \tag{S.37}$$

Then, using (S.36), we have

$$\vec{A}(\vec{r},t) = \frac{1}{4\pi\epsilon_0 c^2}\int_{-d/2}^{d/2} dz\frac{-q_0\,\omega\,\sin[\omega(t-\tilde{r}/c)]}{\tilde{r}}\,\hat{z} \tag{S.38}$$

To first order in d, the potential is just the value at the centre of the above integral:

$$\vec{A}(\vec{r},t) = -\frac{1}{4\pi\epsilon_0 c^2 r}\,p_0\,\omega\,\sin[\omega(t-r/c)]\,\hat{z} \tag{S.39}$$

e) The electric and magnetic fields in terms of the potentials are given by

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t} \tag{S.40}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \tag{S.41}$$

Then, in spherical coordinates, and using the approximation $r \gg \frac{c}{\omega}$, we have

$$\begin{aligned}\vec{\nabla}\Phi &= \frac{\partial\Phi}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial\Phi}{\partial\Theta}\,\hat{\Theta}\\ &= -\frac{p_0\omega}{4\pi\epsilon_o c}\left\{\cos\Theta\left(-\frac{1}{r^2}\,\sin[\omega(t-r/c)] - \frac{\omega}{rc}\,\cos[\omega(t-r/c)]\right)\hat{r}\right.\\ &\left.-\frac{\sin\Theta}{r^2}\,\sin[\omega(t-r/c)]\hat{\Theta}\right\}\\ &\simeq \frac{p_0\omega^2}{4\pi\epsilon_o c^2}\left(\frac{\cos\Theta}{r}\right)\cos[\omega(t-r/c)]\hat{r}\end{aligned} \tag{S.42}$$

$$\frac{\partial\vec{A}}{\partial t} = -\frac{p_0\omega^2}{4\pi\epsilon_0 c^2\,r}\,\cos[\omega(t-r/c)](\cos\Theta\,\hat{r} - \sin\Theta\,\hat{\Theta}) \tag{S.43}$$

$$\begin{aligned}\vec{\nabla}\times\vec{A} &= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\Theta) - \frac{\partial A_r}{\partial\Theta}\right]\hat{\phi}\\ &= -\frac{p_0\omega}{4\pi\epsilon_0 c^2\,r}\left\{\frac{\omega}{c}\,\sin\Theta\,\cos[\omega(t-r/c)] + \frac{\sin\Theta}{r}\,\sin[\omega(t-r/c)]\right\}\hat{\phi}\\ &\simeq -\frac{p_0\omega^2}{4\pi\epsilon_0 c^3}\left(\frac{\sin\Theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\phi}\end{aligned} \tag{S.44}$$

Hence,

$$\vec{E} = -\frac{p_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\sin\Theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\Theta} \tag{S.45}$$

$$\vec{B} = -\frac{p_0\omega^2}{4\pi\epsilon_0 c^3}\left(\frac{\sin\Theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\phi} \tag{S.46}$$

These represents monochromatic waves of frequency ω travelling at the speed of light in the radial direction. The two fields are in phase, orthogonal to each other and to the direction of motion.

Exercise 3. *Spherical waves*

Find the direction of the electric field $\vec{E}$ and the magnetic field $\vec{B}$ of a spherical wave, with respect to the direction of propagation.

Hint. Use Maxwell equations.

Solution. The direction of propagation is $\hat{r}$. Since $\vec{E}$ and $\vec{B}$ satisfy

$$\Box \vec{f}(\vec{r},t) = 0, \tag{S.47}$$

they will have the form

$$\vec{E}(\vec{r},t) = \frac{E(r-ct)}{r}\hat{E} \quad \text{and} \quad \vec{B}(\vec{r},t) = \frac{B(r-ct)}{r}\hat{B}, \tag{S.48}$$

and inserting them in the Maxwell equations in vacuum (using spherical coordinates), we get

$$\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(rE(r-ct)\right)E_r + \frac{1}{r^2}\frac{E(r-ct)}{\tan\theta}E_\theta = 0, \tag{S.49}$$

$$\nabla \cdot \vec{B} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(rB(r-ct)\right)B_r + \frac{1}{r^2}\frac{B(r-ct)}{\tan\theta}B_\theta = 0, \tag{S.50}$$

$$\nabla \times \vec{E} - \frac{\partial B}{\partial t} = \frac{\partial}{\partial t}\frac{B(r-ct)}{r}\hat{B} + \frac{1}{r^2}\frac{E(r-ct)}{\tan\theta}E_\varphi\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}E(r-ct)\left(E_\theta\hat{\varphi} - E_\varphi\hat{\theta}\right) \tag{S.51}$$

$$\tag{S.52}$$

where E_r, E_θ and E_φ are the components of the unit vector $\hat{E}$. Defining $\hat{E}_\perp = \hat{E} - E_r\hat{r}$ and $\hat{B}_\perp = \hat{B} - B_r\hat{r}$, we get

$$\nabla \times \vec{E} - \frac{\partial B}{\partial t} = \frac{\partial}{\partial t}\frac{B(r-ct)}{r}\hat{B} + \frac{1}{r^2}\frac{E(r-ct)}{\tan\theta}E_\varphi\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}E(r-ct)\left(\hat{r}\times\hat{E}_\perp\right) = 0, \tag{S.53}$$

i.e.

$$\frac{\partial}{\partial t}\frac{B(r-ct)}{r}B_r + \frac{1}{r^2}\frac{E(r-ct)}{\tan\theta}E_\varphi = 0, \tag{S.54}$$

$$\frac{\partial}{\partial t}\frac{B(r-ct)}{r}\hat{B}_\perp - \frac{1}{r}\frac{\partial}{\partial r}E(r-ct)\left(\hat{r}\times\hat{E}_\perp\right) = 0, \tag{S.55}$$

It follows, that

$$\hat{B}_\perp \propto \hat{r}\times\hat{E}_\perp, \tag{S.56}$$

i.e.

$$\hat{E}_\perp \perp \hat{B}_\perp. \tag{S.57}$$