Exercise 1. ***Group velocity***

A Gaussian wave package $u(x,t)$ is moving in a dispersive Medium, i.e. ω does not depend linearly on k. At the time $t=0$ we have

$$u(x,t=0) = e^{-\frac{x^2}{2(\Delta x)^2}}, \tag{1}$$

where Δx can be interpret as a measure for the uncertainty. The time dependence is given by

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk u(k) e^{i(kx-\omega(k)t)} \tag{2}$$

where $u(k)$ is the Fourier transform of $u(x,t=0)$.

a) Show by completing the square that the wave package in the momentum space also has a Gaussian profile (put $t=0$). Is there any relation between Δx and Δk? What is the meaning for that?

Solution. We simply Fourier transform the above expression (Eq. (1))

$$\begin{aligned} u(k,0) &= \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{-\frac{x^2}{2(\Delta x)^2}} \mathrm{e}^{-ikx} \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{-\frac{1}{2(\Delta x)^2}\left[x^2+2ikx(\Delta x)^2+\left(ik(\Delta x)^2\right)^2-\left(ik(\Delta x)^2\right)^2\right]} \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{-\frac{1}{2(\Delta x)^2}\left[x+ik(\Delta x)^2\right]^2} \mathrm{e}^{-\frac{k^2(\Delta x)^2}{2}} \mathrm{d}x \\ &= (\Delta x)\, \mathrm{e}^{-\frac{k^2(\Delta x)^2}{2}}. \end{aligned} \tag{S.1}$$

We thus see that $\Delta k = (\Delta x)^{-1}$, i.e. the larger it is in real space, the thinner it is in Fourier space.

b) Let k_0 be the vector for which $u(k_0)$ is maximal. Expand $\omega(k)$ at first order in k around k_0, put it into (2) and show that the maximum of the wave package (up to a phase factor) in the time is moved away by a factor of $v_g\, t$ from the Origin, where the group velocity is given by

$$v_g = \frac{d\omega}{dk}\, |_{k_0} . \tag{3}$$

Solution. Formally, the expansion of $\omega(k)$ around k_0 reads $\omega(k) \approx \omega_0 + (k-k_0)v_g$, where we defined $\omega_0 = \omega(k_0)$ and $v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}\big|_{k_0}$. Plugging it into Eq. (2) leads to

$$\begin{aligned} u(x,t) &\approx \frac{1}{\sqrt{2\pi}} \int u(k) \mathrm{e}^{i\left[kx-\left(\omega_0+(k-k_0)v_g\right)t\right]} \mathrm{d}k \\ &= \frac{1}{\sqrt{2\pi}} \mathrm{e}^{i\left(k_0 v_g-\omega_0\right)t} \int u(k) \mathrm{e}^{ik\left(x-v_g t\right)} \mathrm{d}k \\ &= \mathrm{e}^{i\left(k_0 v_g-\omega_0\right)t} u(x-v_g t, 0). \end{aligned} \tag{S.2}$$

Therefore, we see that the maximum of the wave packet in the time t propagates from $x=0$ to $x=v_g t$.

c) Give the velocity of propagation for the single phases.

Solution. Single phases just propagate with the phase velocity $v_p = \omega(k)/k$.

d) Gives an approximation of how fast is the dispersion of the wave package by finding an expression for the variation of the group velocity within the pulses. Interpret the result! Hint: use the result of part a).

Solution. In the region Δk the group velocity approximately changes by

$$\Delta v_g \approx \frac{\mathrm{d}v_g}{\mathrm{d}k}\Delta k = \frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}(\Delta x)^{-1}\,. \tag{S.3}$$

This means that in a material with $\omega'' > 0$ the wave packet expands faster, the thinner it was at the beginning.

Exercise 2. *Reflection and Transmission*

Two mutually parallel planar surfaces, that are in the same, homogeneous, not magnetised and lossless dielectric with refractive index n, are separated by air ($n = 1$) on the distance d. From the upper half-space, the electromagnetic wave of frequency ω arrives with the angle θ_i to the separating interface. Answer the following questions for the two cases - when the incoming wave is polarised parallel, and perpendicular to the plane of incidence:

a) Calculate the ratio of the reflected wave and that of the wave transmitted through the air to the lower half-space, to the incident wave.

Solution. Let the incident electromagnetic field be given by $\vec{E}_i e^{i\vec{k}\cdot\vec{x}}$, the reflected wave described by $\vec{E}_r e^{i\vec{k}\cdot\vec{x}}$, and the transmitted field to be $\vec{E}_t e^{i\vec{k}'\cdot(\vec{x}-\vec{d})}$. In the middle part we have the two fields $\vec{E}_+ e^{i\vec{k}_0\cdot\vec{x}}$ and $\vec{E}_- e^{i\vec{k}_0'\cdot\vec{x}}$. θ_i and θ_r denotes respectively the incident angle and the light angle inside the part filled with air. Using Snell's law we find

$$n\sin\theta_i = \sin\theta_r \tag{S.4}$$

and since the two external parts are composed by the same dielectric, the transmission angle is the same as the incident angle θ_i. We have

$$\cos\theta_r = \sqrt{1-\sin^2\theta_r} = \sqrt{1-n^2 sin^2\theta_i}. \tag{S.5}$$

If θ_i is larger then the angle of total reflection, $\cos\theta_r$ is imaginary. To express $\vec{E}_t$ and $\vec{E}_r$ as function of $\vec{E}_i$, we use the continuity of the parallel component of $\vec{E}$ and $\vec{H}$. If $\vec{E}$ is perpendicular to the plain that contain the incident and the reflected wave, we have

$$E_{||}: \quad E_i + E_r = E_+ + E_- \tag{S.6}$$

$$E_+ e^{i\varphi} + E_- e^{-i\varphi} = E_t \tag{S.7}$$

$$H_{||}: \quad n(E_i - E_r)\cos\theta_i = (E_+ - E_-)\cos\theta_r \tag{S.8}$$

$$(E_+ e^{i\varphi} - E_- e^{-i\varphi})\cos\theta_r = nE_t\cos\theta_i, \tag{S.9}$$

where $\varphi = \vec{k}_0\cdot d = k_0 d\cos\theta_r = c^{-1}\omega d\cos\theta_r = c^{-1}\omega d\sqrt{1-n^2 sin^2\theta_i}$.

It follows that

$$E_+ = \frac{1}{2}\left(E_i(1+\alpha) + E_r(1-\alpha)\right) \tag{S.10}$$

$$E_- = \frac{1}{2}\left(E_i(1-\alpha) + E_r(1+\alpha)\right), \tag{S.11}$$

and

$$E_+ = \frac{1}{2}e^{-i\varphi}E_t(1+\alpha) \tag{S.12}$$

$$E_- = \frac{1}{2}e^{i\varphi}E_t(1-\alpha), \tag{S.13}$$

where

$$\alpha = n\frac{\cos\theta_i}{\cos\theta_r} = n\frac{\cos\theta_i}{\sqrt{1-n^2 sin^2\theta_i}}. \tag{S.14}$$

Combining the equations, we find

$$\frac{E_t}{E_i} = \frac{4\alpha}{(1+\alpha)^2 e^{-i\varphi} - (1-\alpha)^2 e^{i\varphi}} = \frac{2\alpha}{2\alpha\cos\varphi - i(1+\alpha^2)\sin\varphi} \tag{S.15}$$

$$\frac{E_r}{E_i} = \frac{(1-\alpha^2)(e^{i\varphi}-e^{-i\varphi})}{(1+\alpha)^2 e^{-i\varphi} - (1-\alpha)^2 e^{i\varphi}} = \frac{i(1-\alpha^2)\sin\varphi}{2\alpha\cos\varphi - i(1+\alpha^2)\sin\varphi}. \tag{S.16}$$

If θ_i is smaller than the total reflection angle, α and φ are real and the transmission and reflection coefficients are given by

$$T = \left|\frac{E_t}{E_i}\right|^2 = \frac{4\alpha^2}{4\alpha^2\cos^2\varphi + (1+\alpha^2)^2\sin^2\varphi} = \frac{4\alpha^2}{4\alpha^2 + (1-\alpha^2)^2\sin^2\varphi} \tag{S.17}$$

$$R = \left|\frac{E_r}{E_i}\right|^2 = \frac{(1-\alpha^2)^2\sin^2\varphi}{4\alpha^2\cos^2\varphi + (1+\alpha^2)^2\sin^2\varphi} = \frac{(1-\alpha^2)^2\sin^2\varphi}{4\alpha^2 + (1-\alpha^2)^2\sin^2\varphi}. \tag{S.18}$$

As expected the relation $T + R = 1$ holds. T oscillates between $(2\alpha/(1+\alpha^2))^2$ and 1, when changing the wavelength.

If $\vec{E}$ is parallel to the plain that contains the incident and the reflected wave we find

$$E_{||}: \quad (E_i - E_r)\cos\theta_i = (E_+ - E_-)\cos\theta_r \tag{S.19}$$

$$(E_+ e^{i\varphi} - E_- e^{-i\varphi})\cos\theta_r = E_t\cos\theta_i \tag{S.20}$$

$$H_{||}: \quad n(E_i + E_r) = E_+ + E_- \tag{S.21}$$

$$(E_+ e^{i\varphi} + E_- e^{-i\varphi}) = nE_t. \tag{S.22}$$

It follows that the same relations holds, if we replace $E_\pm \to n^{-1}E_\pm$ and

$$\alpha \to \beta = \frac{\cos\theta_i}{n\sqrt{1-n^2\sin^2\theta_i}}. \tag{S.23}$$

The transmission and reflection coefficients are also the same replacing $\alpha \to \beta$.

b) Sketch for incident angles θ_i larger than the critical angle for the total reflection, the ratio of the transmitted to incident wave as a function of the layer thickness d measured in the units of the wavelength.

Solution. We look at the case when $\vec{E}$ is perpendicular to the plane of incidence. As θ_i is bigger than the critical angle, α and ϕ are imaginary and we define:

$$\alpha = i\gamma \tag{S.24}$$

$$\phi = i\eta$$

Hence we have:

$$T = |\frac{E_t}{E_i}|^2 = \frac{4\gamma^2}{4\gamma^2 + (1+\gamma^2)^2\sinh^2\eta} \tag{S.25}$$

$$R = |\frac{E_r}{E_i}|^2 = \frac{(1+\gamma^2)^2\sinh^2\eta}{4\gamma^2 + (1+\gamma^2)^2\sinh^2\eta} \tag{S.26}$$

where

$$\gamma = -\frac{n\cos\theta_i}{\sqrt{n^2\sin{\theta_i}^2 - 1}} \tag{S.27}$$

$$\eta = \frac{\omega d\sqrt{n^2\sin{\theta_i}^2 - 1}}{c} \tag{S.28}$$

In this case there are no oscillations in the transmitted wave, but the exponential damping. $T \to 1$ when $d \to 0$ (or $\eta \to 0$ and $T \to 0$ when $d \to \infty$ (or $\eta \to \infty$)).

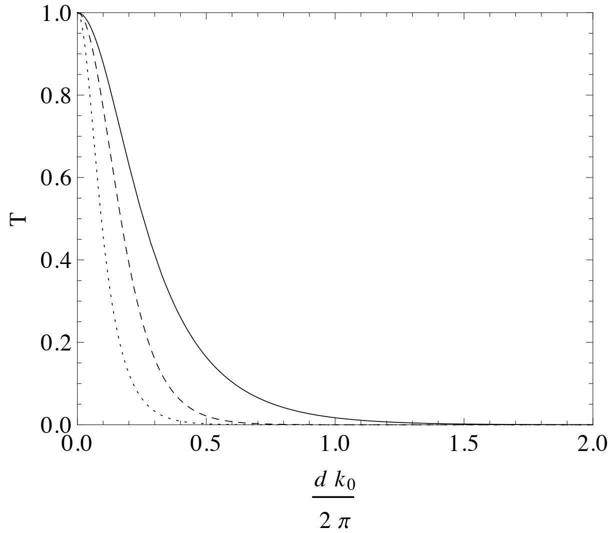


Figure 1: transmitted wave for the angles $\theta_i = 45$ (full curve), $\theta_i = 60$ (lined curve) and $\theta_i = 70$ (pointed curve).

Exercise 3. *Photonic Band Gap Material*

Photonic Band Gap Materials are crystals in which some frequencies of light cannot propagate. The term *band gap* reminds one of the semiconductors, where electrons below certain energies are also not free. In this exercise, you will see a simple model of a Photonic Band Gap material.

a) Derive the generalised wave equations satisfied by the electric field $\vec{E}(\vec{r},t)$ in a non-magnetic matter when the permitivity is a function of position, $\epsilon = \epsilon(\vec{r})$. Later on specialize to the case when $\epsilon(\vec{r}) = \epsilon(z)$ and also $\vec{E}(\vec{r},t) = \hat{x}E(z,t)$.

Solution. Maxwell's equations in a non-magnetic matter (without free charges and currents) take the form:

$$\nabla \cdot \vec{D} = 0 \tag{S.29}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{S.30}$$

$$\nabla \cdot \vec{B} = 0 \tag{S.31}$$

$$\nabla \times \vec{H} = \frac{\partial D}{\partial t} \tag{S.32}$$

where $\vec{D} = \epsilon\vec{E}$ and $\vec{H} = \mu_0\vec{B}$. Take the rotation of the equation (S.30) and use other equations as well as definitions of $\vec{D}$ and $\vec{H}$ to obtain:

$$\nabla(\nabla \cdot \vec{E}) = \nabla^2\vec{E} - \mu_0\epsilon\frac{\partial^2\vec{E}}{\partial t}. \tag{S.33}$$

Now specify to $\vec{E}(\vec{r},t) = \hat{x}E(z,t)$ and $\epsilon(\vec{r}) = \epsilon(z)$ to finally get:

$$\partial_z^2 E(z,t) = \mu_0\epsilon(z)\partial_t^2 E(z,t) \tag{S.34}$$

b) Let $E(z,t) = E(z)exp(-i\omega t)$ and also let $\epsilon(z) = \epsilon_0[1+\alpha\cos(2k_0 z)]$. Show that the Fourier components $\hat{E}(k)$ of the electric field satify the coupled set of the linear equations:

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\hat{E}(k) = \frac{\omega^2\alpha}{2c^2}\left[\hat{E}(k-2k_0) + \hat{E}(k+2k_0)\right]. \tag{4}$$

Solution. Plugging the wave expansion of $E(z,t)$ ($E(z,t) = E(z)exp(-i\omega t)$) and the form of $\epsilon(z)$ one obtains:

$$\partial_z^2 E(z) = -\frac{\omega^2}{c^2}(1+\alpha\cos(2k_0 z)E(z) \tag{S.35}$$

where we used that $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$. Now plug in the Fourier expansion of $E(z)$:

$$E(z) = \int \hat{E}(k)e^{ikz}dk \tag{S.36}$$

into (S.35):

$$\int dk(-k^2)\hat{E}(k)e^{ikz} = -\frac{\omega^2}{c^2}\int dk\hat{E}(k)e^{ikz} - \frac{\alpha\omega^2}{2c^2}\int dk\hat{E}(k)\left(e^{iz(k+2k_0)} + e^{iz(k-2k_0)}\right). \tag{S.37}$$

Simpliify, play with the integration variable (i.e. shift $k \to k \pm 2k_0$) to finally get the desired result

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\hat{E}(k) = \frac{\omega^2\alpha}{2c^2}\left[\hat{E}(k-2k_0) + \hat{E}(k+2k_0)\right]. \tag{S.38}$$

c) Suppose $\alpha << 1$ and focus on values of k in the vicinity of k_0, i.e. set $k = k_0 + q$, where $|q| << k_0$. Show that the Fourier components $\hat{E}(q+k_0)$ and $\hat{E}(q-k_0)$ are larger than all the others and therefore the 2×2 eigenvalue problem determines the dispersion relation. *Hint*: The wave frequency cannot differ greatly from its $\alpha = 0$ value in the limit considered.

Solution. In the $\alpha = 0$ limit the dispersion relation is just:

$$k^2 = \frac{\omega^2}{c^2} \tag{S.39}$$

so if we set $k = k_0 + q$, then $\omega \approx c(k_0+q)$. Thus setting $k = k_0 + q$ in the equation (S.38) yields:

$$\left((q+k_0)^2 - \frac{\omega^2}{c^2}\right)\hat{E}(q+k_0) = \frac{\alpha\omega^2}{2c^2}\left\{\hat{E}(q-k_0) + \hat{E}(q+3k_0)\right\}. \tag{S.40}$$

Now, ω is close to the $\alpha = 0$ value ($\omega \approx c(k_0+q)$), and so the LHS is small regardless of the magnitude of $\hat{E}(q+k_0)$. Since also α is small we are good ;). We now fix $\omega \approx c(k_0+q)$ and take $k = q + 3k_0$. Then the equation (S.38) takes the form:

$$\left((q+3k_0)^2 - \frac{\omega^2}{c^2}\right)\hat{E}(q+3k0) = \frac{\alpha\omega^2}{2c^2}\left\{\hat{E}(q+k_0) + \hat{E}(q+5k_0)\right\} \tag{S.41}$$

But now the terms in the bracket on the LHS are not necessarily small, since still we have that $\omega = c(k_0+q) + O(\alpha)$. As the RHS is small (due to the appearance of α) we see that $\hat{E}(q+3k_0)$ must be small. Similarly for $\hat{E}(q-k_0)$ (not small) and $\hat{E}(q-3k_0)$ (small).

Thus eventually the dominating equations in the limits considered are the following two:

$$\hat{E}(k_0+q)\left((k_0+q)^2 - \frac{\omega^2}{c^2}\right) = \frac{\alpha\omega^2}{2c^2}\hat{E}(q-k_0) \tag{S.42}$$

$$\hat{E}(q-k_0)\left((q-k_0)^2 - \frac{\omega^2}{c^2}\right) = \frac{\alpha\omega^2}{2c^2}\hat{E}(q+k_0). \tag{S.43}$$

d) Solve the eigenvalue problem to find $\omega(k_0, q)$. Study it's behavior at $q = 0$ (also remember that α is a small parameter.). Sketch the complete dispersion curve and show that there is a range of frequencies - called a photonic band gap - where no waves occur.

Solution. Ok, so since the above system of the equations is homogeneous, the non-trival solution exists only if the determinant of the coefficient matrix has to vanish. This condition results in the following equation:

$$\left(1-\frac{1}{4}\alpha^2\right)\left(\frac{\omega^2}{c^2}\right)^2-\left((q-k_0)^2+(q+k_0)^2\right)\frac{\omega^2}{c^2}+(q-k_0)^2(q+k_0)^2=0 \tag{S.44}$$

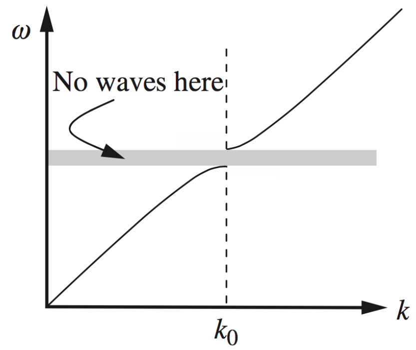
It has the following solutions:

$$\frac{\omega_\pm^2}{c^2}=\frac{q^2+k_0^2\pm\sqrt{(q^2+k_0^2)^2-(q^2-k_0^2)^2(1-\alpha^2/4)}}{1-\alpha^2/4} \tag{S.45}$$

We can now set $q=0$ to obtain:

$$\omega_\pm = ck_0\sqrt{\frac{1}{1\mp\frac{\alpha}{2}}} \tag{S.46}$$

so the frequency will jump at $k=k_0$ from ω_- to ω_+. One can sketch the dispersion relation for a non-zero q to obtain the following graph:



The function is $\omega(k)=ck$ except for a vicinity of $k=k_0$ where the jump occurs. Thus there is a range of frequencies where no wave can propagate. A wave with a frequency inside that range, incident on the crystal with the $\epsilon(z)$ as considered, will be totally reflected from it.