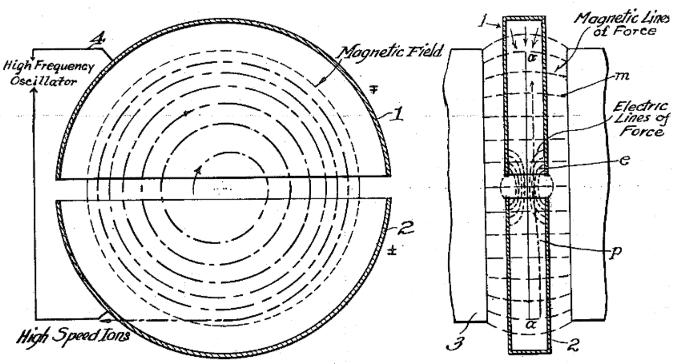
Exercise 1. ***Cyclotron***

A cyclotron is a particle accelerator for the charged particles. Magnetic field keeps the particles in a circular motion, and the energy for acceleration is provided by the electric field. A simple cyclotron consists of the two half-cylinders, placed next two each other, with two gaps in between (only places where electric field exists). Magnetic field is constant and is perpendicular to the motion of the particles, while the electric field in the gaps has the direction of the motion of the particles, or the opposite one. Inside the cylinders, the electric field is zero. A charged particle starts out in the central point and follows a circular path due to the magnetic field. An alternating voltage (with a potential difference U) is applied across the two gaps, and when the particle crosses the gap, the voltage acts to accelerate it.



a) Show by using the four momentum and the integration of the Lorentz force, that the energy gain of a particle (charge q) when crossing the gap, is given by the electric potential difference: $\Delta E = qU$.

Solution. We know that

$$\frac{d}{dt}p_\mu p^\mu = \frac{d}{dt}m^2c^2 = 0 \tag{S.1}$$

Hence (as in Sheet9, Q1), we get that

$$\mathcal{E}\frac{d\mathcal{E}}{dt} = c^2\vec{p}\frac{d}{dt}\vec{p} \tag{S.2}$$

Hence, using the fact that $\frac{d}{dt}\vec{p} = \vec{F} = q\vec{E}$ as the magnetic field is perpendicular to the velocity, and $\vec{p} = \mathcal{E}\vec{v}/c^2$, we have that

$$\frac{d}{dt}\mathcal{E} = q\vec{E}\vec{v} \tag{S.3}$$

Now we know that $\vec{E}\vec{d} = U$, hence integrating the energy over time we have that $\Delta\mathcal{E} = qU$.

b) How should the cyclotron frequency (frequency of the particles) ω_n change with each turn n in the case of uniform B, so that we maintain the synchronisation between the cyclotron and orbital movement of accelerated particles? How often does the polarity of the electric field, induced by the potential in the gaps, has to change, for the particle to keep accelerating? Calculate the frequency of this change in comparison to the frequency of the particle due to the magnetic field (cyclotron frequency). How will the particle path look like? (*Hint: compare this to the Sheet 8, Q1, and Sheet 10, Q2*).

Solution. The frequency only due to the magnetic field is

$$\omega = \frac{qB}{\gamma m} \tag{S.4}$$

As we showed that the change in the electric field in the gaps induces the change in the energy, this will affect γ factor, and hence the frequency. We have that $\gamma_n = \frac{\mathcal{E}_n}{mc^2}$, hence

$$\omega_n = \frac{qBc^2}{\mathcal{E}_n} \tag{S.5}$$

with $\mathcal{E}_n = \mathcal{E}_0 + 2nqU + \mathcal{E}_{kin,0}$, , with $n = \frac{1}{2}, 1, \ldots.$ Polarity of the electric field should change each half circle, and the frequency should equal the cyclotron frequency. E.g. if $E = \cos(\omega t)$, then $\omega = \omega_n$.The path of the particle will be a spiral (each time a radius of the circle will increase due to the acceleration induced by the electric field, until the maximum radius is reached).

c) What is the maximum velocity of the proton for the maximum radius ρ_0?

Solution. We know that for the given radius ρ, $\omega = \frac{v}{\rho}$. Hence

$$v = \rho_0 \omega = \rho_0 \frac{qB\sqrt{1 - \frac{v^2}{c^2}}}{m} \tag{S.6}$$

Solving this equation in v:

$$v_{max} = \frac{c\rho_0 qB}{\sqrt{\rho_0^2 q^2 B^2 + m^2 c^2}} \tag{S.7}$$

d) What is the kinetic energy of the particle?

Solution. $\mathcal{E}_{kin} = mc^2(\gamma(\vec{v}) - 1)$

e) If the maximum voltage across the gap is V_0, how many full circles does the proton makes before it reaches its maximum energy?

Solution. Number of circles equals the total gain in electric energy divided by the gain in each circle $(2qV_0)$. Using formula for the maximal velocity:

$$n = \frac{\mathcal{E}_{kin,max} - \mathcal{E}_{kin,0}}{2\Delta\mathcal{E}} =$$

$$\frac{\mathcal{E}_{kin,max} - \mathcal{E}_{kin,0}}{2qV_0} = \frac{mc^2(\gamma_{max} - 1) - mc^2(\gamma_0 - 1)}{2qV_0} =$$

$$\frac{mc^2}{2qV_0\sqrt{\frac{m^2c^2}{\rho^2 B^2 q^2 + m^2 c^2}}} - \frac{mc^2\gamma_0}{2qV_0} = \frac{c\sqrt{\rho^2 B^2 q^2 + m^2 c^2}}{2qV_0} - \frac{mc^2\gamma_0}{2qV_0}$$