Exercise 1. ***Energy loss due to radiation of accelerating charges***

In this exercise, we will look at the radiation of accelerated charges in the relativistic limit. The energy loss associated with this is important especially in particle accelerators. Here we will consider both linear and circular accelerators.

1. Show that the power radiated from an accelerated electron of charge e and mass m is given by
$$P = \frac{2ke^2}{3c}\gamma^6[|\dot{\vec{v}}|^2 - |\vec{v}\times\dot{\vec{v}}|^2/c^2], \tag{1}$$
where $k = \frac{1}{4\pi\epsilon_0}$ and $\dot{\vec{v}} = d\vec{v}/dt$.
To this end consider the Lorentz invariant generalisation of the Larmor's formula
$$P = -\frac{2ke^2}{3m^2c^3}\left(\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right) \tag{2}$$
and rewrite it in terms of its components. Upon using $E = \gamma mc^2$ and $\vec{p} = \gamma m\vec{v}$ and after some algebra you should obtain the desired result.

2. Assume now that the acceleration is in the direction of motion as in a linear accelerator. Show that in the relativistic limit as $v \to c$ the power radiated from an accelerated electron can be written as
$$P = \frac{2ke^2}{3m^2c^4}\frac{dE}{dx}\frac{dE}{dt} \tag{3}$$
with dx being the infinitesimal distance travelled by the electron.

   For an energy gain of $10MeV/m$ through acceleration for electrons with $mc^2 = 0.5MeV$, what is the fraction of the radiated power to the energy gain per unit time? Is it significant?

3. For a circular accelerator, the tangential acceleration is negligible relative to the orthogonal acceleration, and so we can write $|\dot{v}| = v\,\omega$ with ω the angular frequency. Starting from Equation 2 show that in this case the power can written as
$$P = \frac{2ke^2c}{3\rho^2}\left(\frac{v}{c}\right)^4\gamma^4 \tag{4}$$
where ρ is the orbit radius and $\omega = v/\rho$.
Now what is the energy loss during one period as $v \to c$? Calculate this explicitly assuming a typical energy of $E = \gamma mc^2 = 10GeV$ and radius $\rho = 10m$ of an electron-synchrotron. Observe how this is much more significant than the effect of radiation in the case of linear acceleration.

Solution. Throughout the solution, we shall use $\vec{\beta} = \vec{v}/c$.

1. $E = \gamma mc^2$ and $\vec{p} = \gamma m\vec{v}$. We write
$$P = -\frac{2e^2}{3m^2c^3}\left(\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right) = P = -\frac{2e^2}{3m^2c^3}\left(\frac{dp_\mu}{dt}\frac{dt}{d\tau}\frac{dp^\mu}{dt}\frac{dt}{d\tau}\right) = -\frac{2e^2}{3m^2c^3}\left(\frac{dp_\mu}{dt}\right)^2\left(\frac{dt}{d\tau}\right)^2 \tag{S.1}$$

$$\begin{aligned}
\frac{dp_\mu}{dt} =& \frac{d}{dt}\left(\gamma mc, \gamma mc\vec{\beta}\right) = \frac{d}{dt}\gamma mc\left(1, \vec{\beta}\right) && \text{(S.2)}\\
=& \gamma mc\left(0, \dot{\vec{\beta}}\right) + \dot{\gamma} mc\left(1, \vec{\beta}\right) && \text{(S.3)}\\
=& \gamma mc\left(0, \dot{\vec{\beta}}\right) + \gamma^3\,\vec{\beta}\cdot\dot{\vec{\beta}}\,mc\left(1, \vec{\beta}\right) && \text{(S.4)}\\
=& \left(\gamma^3\,\vec{\beta}\cdot\dot{\vec{\beta}}\,mc, \gamma mc\dot{\vec{\beta}} + \gamma^3\,\vec{\beta}\cdot\dot{\vec{\beta}}\,mc\,\vec{\beta}\right) && \text{(S.5)}
\end{aligned}$$

where in the third line we used

$$\dot{\gamma} = \frac{d}{dt}\frac{1}{\sqrt{1-\beta^2}} = \frac{\vec{\beta}\frac{d\vec{\beta}}{dt}}{(1-\beta^2)^{3/2}} = \gamma^3\,\vec{\beta}\cdot\dot{\vec{\beta}}$$

Then we have

$$\begin{aligned}
\left(\frac{dp_\mu}{dt}\right)^2 =& \left(\gamma^3\,\vec{\beta}\cdot\dot{\vec{\beta}}\,mc, \gamma mc\dot{\vec{\beta}} + \gamma^3\,\vec{\beta}\cdot\dot{\vec{\beta}}\,mc\,\vec{\beta}\right)^2 && \text{(S.6)}\\
=& \gamma^6\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 m^2c^2 - \gamma^2 m^2c^2\,\dot{\vec{\beta}}^2 - 2\gamma^4 m^2c^2\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 - \gamma^6 m^2c^2\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)\left(\vec{\beta}\cdot\vec{\beta}\right) && \text{(S.7)}\\
=& \gamma^6 m^2c^2\left(\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 - \gamma^{-4}\dot{\vec{\beta}}^2 - 2\gamma^{-2}\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 - \left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2\left(\vec{\beta}\cdot\vec{\beta}\right)\right) && \text{(S.8)}\\
=& \gamma^6 m^2c^2\left(\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 - \left(1-\beta^2\right)^2\dot{\vec{\beta}}^2 - 2\left(1-\beta^2\right)\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 - \left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2\left(\vec{\beta}\cdot\vec{\beta}\right)\right) && \text{(S.9)}\\
=& \gamma^6 m^2c^2\left(\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 - \dot{\vec{\beta}}^2 + 2\beta^2\dot{\vec{\beta}}^2 - \beta^4\dot{\vec{\beta}}^2 - 2\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 + 2\beta^2\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 - \beta^2\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2\right) && \text{(S.10)}\\
=& \gamma^6 m^2c^2\left(-\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 - \dot{\vec{\beta}}^2 + 2\beta^2\dot{\vec{\beta}}^2 - \beta^4\dot{\vec{\beta}}^2 + \beta^2\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2\right) && \text{(S.11)}\\
=& \gamma^6 m^2c^2\left(-\left(1-\beta^2\right)\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 - \dot{\vec{\beta}}^2 + \beta^2\dot{\vec{\beta}}^2 + \beta^2\dot{\vec{\beta}}^2 - \beta^4\dot{\vec{\beta}}^2\right) && \text{(S.12)}\\
=& \gamma^6 m^2c^2\left(-\left(1-\beta^2\right)\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 - \left(1-\beta^2\right)\dot{\vec{\beta}}^2 + \left(1-\beta^2\right)\beta^2\dot{\vec{\beta}}^2\right) && \text{(S.13)}\\
=& \gamma^4 m^2c^2\left(-\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 - \dot{\vec{\beta}}^2 + \beta^2\dot{\vec{\beta}}^2\right) && \text{(S.14)}\\
=& \gamma^4 m^2c^2\left(|\beta\times\dot{\vec{\beta}}|^2 - \dot{\vec{\beta}}^2\right) && \text{(S.15)}
\end{aligned}$$

where in the last line we used the fact that

$$|\beta\times\dot{\vec{\beta}}|^2 = \beta^2\dot{\vec{\beta}}^2 - \left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2$$

Plugging in everything together with

$$\left(\frac{dt}{d\tau}\right)^2 = \gamma^2$$

we find

$$P = \frac{2ke^2}{3c}\gamma^6\left(\dot{\vec{\beta}}^2 - |\beta\times\dot{\vec{\beta}}|^2\right)$$

2. Assume without loss of generality that the motion is in the x axis. Now, using $d\tau = \frac{1}{\gamma}dt$ and $\frac{dp^0}{d\tau} = \beta\frac{dp^1}{d\tau}$, which comes from $p_\mu p^\mu = mc^2$ giving $p^0\frac{dp^0}{d\tau} = p^1\frac{dp^1}{d\tau}$, we get

$$-\left(\frac{dp}{d\tau}, \frac{dp}{d\tau}\right) = \left(\frac{dp^1}{d\tau}\right)^2\left(1-\beta^2\right) = \left(\frac{dp^1}{dt}\right)^2. \tag{S.16}$$

Furthermore,

$$\frac{dp^1}{dt} = \frac{1}{v}\frac{dE}{dt} = \frac{dE}{dx} \tag{S.17}$$

and so

$$\frac{P}{dE/dt} = \frac{2ke^2}{3m^2c^3}\frac{1}{v}\frac{dE}{dx}, \tag{S.18}$$

and so for $\beta\to 1$, $P \to \frac{2ke^2}{3m^2c^4}\frac{dE}{dx}$. For electrons with $mc^2 = 0.5MeV$ and an energy gain of $10MeV/m$ we obtain $P \approx 3.84\times 10^{-14}dE/dt$, which is negligibly low compared to the energy gain per unit time due to the acceleration.

3. For a circular accelerator with frequency ω, we note that $\dot{\beta} = \beta\omega$ and $\omega = c\beta/\rho$, where ρ is the radius of the circle.
We now use Equation 1 and calculate

$$P = \frac{2ke^2}{3c}(1-\beta^2)^{-3}[\omega^2\beta^2 - \omega^2\beta^4] \tag{S.19}$$
$$= \frac{2ke^2}{3c}(1-\beta^2)^{-2}\beta^2\omega^2. \tag{S.20}$$

During one period (the time $2\pi/\omega$, the energy loss is hence given by

$$\Delta E = \frac{4\pi ke^2}{3c}(1-\beta^2)^{-2}\beta^2\omega \tag{S.21}$$
$$= \frac{4\pi ke^2}{3\rho}(1-\beta^2)^{-2}\beta^3, \tag{S.22}$$

which for $\beta \to 1$ becomes $\approx \frac{4\pi ke^2}{3\rho}(1-\beta^2)^{-2}$. With $E = mc^2(1-\beta^2)^{-1/2}$ this is just

$$\Delta E = \frac{4\pi ke^2}{3\rho}\left(\frac{E}{mc^2}\right)^4. \tag{S.23}$$

Now, for $E = 10GeV \approx 1.956 \times 10^4 mc^2$ and $\rho = 10m$, we find $\Delta E \approx 1.41 \times 10^{-11}J$. This is significant in comparison with the energy provided during one period (remember that $1MeV \approx 1.6 \times 10^{-13}J$): for circular accelerators the loss due to radiation is one of the most important limiting factors for the achievable particle energy.

Exercise 2. *The Lorentz transformation of acceleration*

In this exercise we will find how the acceleration transforms from one frame to another under Lorentz transformations.

Consider a reference frame $\mathcal{S}'$ that moves away from another frame $\mathcal{S}$ with a velocity $\mathbf{v}_{S'}$. In frame $\mathcal{S}'$ there is a particle moving with a velocity $\mathbf{u}'$ and an acceleration $\mathbf{a}'$.

Show that the acceleration transformed into the frame $\mathcal{S}$ has the following components:

$$\mathbf{a}_{\|} = \frac{\left(1 - \frac{v_{S'}^2}{c^2}\right)^{3/2}}{\left(1 + \frac{\mathbf{v}_{S'}\cdot\mathbf{u}'}{c^2}\right)^3} \quad \mathbf{a}'_{\|} \tag{5}$$

$$\mathbf{a}_{\perp} = \frac{\left(1 - \frac{v_{S'}^2}{c^2}\right)}{\left(1 + \frac{\mathbf{v}_{S'}\cdot\mathbf{u}'}{c^2}\right)^3} \left(\mathbf{a}'_{\perp} + \frac{\mathbf{v}_{S'}}{c^2} \times (\mathbf{a}' \times \mathbf{u}')\right) \tag{6}$$

where $\mathbf{a}_{\|,\perp}$ are the acceleration parallel/perpendicular to the direction of motion.

Solution. We choose the direction of the movement of the reference frame such that $\mathbf{v}_{S'} = v_{S'}\hat{x}$.

$$dt = \gamma\left(dt' + \frac{v_{S'}}{c^2}dx'\right) \tag{S.24}$$

Then we have

$$\frac{dx}{dt} = \frac{\gamma(dx' + v_{S'}dt')}{\gamma\left(dt' + \frac{v_{S'}}{c^2}dx'\right)} = \frac{dx'/dt' + v_{S'}}{1 + \frac{v_{S'}\frac{dx'}{dt'}}{c^2}} = \frac{u'_x + v_{S'}}{1 + \frac{v_{S'}u'_x}{c^2}} \tag{S.25}$$

Since u_x is the parallel component of the velocity we can generalise as:

$$\mathbf{u}_\parallel = \frac{u'_\parallel + v_{S'}}{1 + \frac{v_{S'} u'_\parallel}{c^2}} \tag{S.26}$$

Together with the perpendicular component we can write

$$\mathbf{u}_\parallel = \frac{\mathbf{u}'_\parallel + \mathbf{v}_{S'}}{1 + \frac{\mathbf{v}_{S'} \cdot \mathbf{u}'}{c^2}}, \quad \mathbf{u}_\perp = \frac{\mathbf{u}'_\perp}{\gamma \left(1 + \frac{\mathbf{v}_{S'} \cdot \mathbf{u}'}{c^2}\right)} \tag{S.27}$$

Then

$$\mathbf{a}_\parallel = \frac{d\mathbf{u}_\parallel}{dt} = \frac{\left(1 + \frac{\mathbf{v}_{S'} \cdot \mathbf{u}'}{c^2}\right) d\mathbf{u}'_\parallel - \left(\mathbf{u}'_\parallel + \mathbf{v}_{S'}\right) \frac{v_{S'}}{c^2} d\mathbf{u}'_\parallel}{\left(1 + \frac{\mathbf{v}_{S'} \cdot \mathbf{u}'}{c^2}\right)^2 \gamma \left(dt' + \frac{v_{S'}}{c^2} dx'\right)} \tag{S.28}$$

Using $\mathbf{a}'_\parallel = d\mathbf{u}'_\parallel / dt'$ we arrive at

$$\mathbf{a}_\parallel = \frac{\mathbf{a}'_\parallel \left(1 - \frac{v_{S'}^2}{c^2}\right)}{\gamma \left(1 + \frac{v_{S'} u'_\parallel}{c^2}\right)^3} = \frac{\mathbf{a}'_\parallel \left(1 - \frac{v_{S'}^2}{c^2}\right)^{3/2}}{\left(1 + \frac{v_{S'} u'_\parallel}{c^2}\right)^3} \tag{S.29}$$

where in the third line γ is simplified with the numerator.

The perpendicular part is

$$\mathbf{a}_\perp = \frac{d\mathbf{u}_\perp}{dt} = \frac{d\left(\frac{\mathbf{u}'_\perp}{\gamma\left(1 + \frac{\mathbf{v}_{S'} \cdot \mathbf{u}'}{c^2}\right)}\right)}{\gamma \left(dt' + \frac{v_{S'}}{c^2} dx'\right)} \tag{S.30}$$

$$d\mathbf{u}_\perp = \frac{d\mathbf{u}'_\perp}{\gamma \left(1 + \frac{\mathbf{v}_{S'} \cdot \mathbf{u}'}{c^2}\right)} - \frac{u'_\perp \frac{\mathbf{v}_{S'} \cdot d\mathbf{u}'}{c^2}}{\left(1 + \frac{\mathbf{v}_{S'} \cdot \mathbf{u}'}{c^2}\right)^2} \tag{S.31}$$

Plugging in $\mathbf{v}_{S'} \cdot d\mathbf{u}' = v_{S'} du'_\parallel$

$$\mathbf{a}_\perp = \frac{\left(1 + \frac{\mathbf{v}_{S'} \cdot \mathbf{u}'}{c^2}\right) d\mathbf{u}'_\perp - \frac{v'_S}{c^2} \mathbf{u}'_\perp d\mathbf{u}'_\parallel}{\gamma^2 \, dt' \left(1 + \frac{\mathbf{v}_{S'} \cdot \mathbf{u}'}{c^2}\right)^3} \tag{S.32}$$

Since $\mathbf{a}'_\perp = \frac{d\mathbf{u}'_\perp}{dt'}$ and $\mathbf{a}'_\parallel = \frac{d\mathbf{u}'_\parallel}{dt'}$

$$\mathbf{a}_\perp = \frac{\left(1 - \frac{v_{S'}^2}{c^2}\right)}{\left(1 + \frac{\mathbf{v}_{S'} \cdot \mathbf{u}'}{c^2}\right)^3} \left(\mathbf{a}'_\perp + \frac{\mathbf{v}_{S'} \cdot \mathbf{u}'}{c^2} \mathbf{a}'_\perp - \frac{\mathbf{v}_{S'} \cdot \mathbf{a}'}{c^2} \mathbf{u}'_\perp\right) \tag{S.33}$$

$$= \frac{\left(1 - \frac{v_{S'}^2}{c^2}\right)}{\left(1 + \frac{\mathbf{v}_{S'} \cdot \mathbf{u}'}{c^2}\right)^3} \left(\mathbf{a}'_\perp + \frac{\mathbf{v}_{S'}}{c^2} \times \left(\mathbf{a}' \times \mathbf{u}'\right)\right) \tag{S.34}$$

Exercise 3. *Twin paradox revisited*

Consider a spaceship leaving the Earth today. On board there is one of two twins born in 1995. The other remains on Earth. Assume that the spaceship, in its own reference frame, has an acceleration $a' = 30$ m/s^2. It accelerates for 5 years (time in its own frame) in a straight-line path, decelerates at the same rate for 5 years, turns around, accelerates for 5 years and decelerates again for 5 years. When the astronaut returns back on Earth he is 40 years old.

1. How old is the twin that stayed on Earth?

Solution. Due to the symmetry of the problem we only consider the first piece (initial acceleration). The final result will be four times the initial piece.

Using the results of the previous exercise we can find the acceleration as seen from the Earth. The constant acceleration a' (in the reference frame of the space ship) is transformed to

$$a_{\|} = \frac{a'_{\|}\left(1 - v^2/c^2\right)^{3/2}}{\left(1 + vu'_{\|}/c^2\right)^3} = a'\left(1 - v^2/c^2\right)^{3/2}, \tag{S.35}$$

in the Earth's reference frame, where v is the velocity of the spaceship. This expression can be rewritten as

$$a'\,dt = \left(1 - v^2/c^2\right)^{-3/2} dv, \tag{S.36}$$

which, integrated, gives the velocity as a function of time

$$v(t) = \frac{a't}{\sqrt{1 + (a't/c)^2}}. \tag{S.37}$$

Now we look at the relation between the times in the two reference frames. Since in the primed frame the space ship is at rest we have $dt = \gamma(t)dt'$. Using Eq. (S.37) for $v(t)$ leads to

$$dt' = dt/\gamma(t) = \frac{dt}{\sqrt{1 + (a't/c)^2}}. \tag{S.38}$$

Lastly we integrate Eq. (S.38) obtaining

$$t'_f - t'_i = \frac{c}{a'}\left(\sinh^{-1}\left(\frac{a't_f}{c}\right) - \sinh^{-1}\left(\frac{a't_i}{c}\right)\right). \tag{S.39}$$

For the first piece of travel we thus obtain the following time on Earth

$$t_f = \frac{c}{a'}\sinh\left(\frac{a't'_f}{c}\right) \approx 1.1 \times 10^6 \text{ years}. \tag{S.40}$$

The total time elapsed on Earth during the journey is $t_{\text{tot}} \approx 4 \cdot 1.1 \times 10^6$ years $= 4.4 \times 10^6$ years.

2. How far away from the Earth did the spaceship travel as seen from the Earth's reference frame?

Solution. Here we consider again only the first piece. The final distance will be twice that covered in the first piece.

From Earth we have $dx = v(t)dt$, where $v(t)$ is given by (S.37). Integration from $t_i = 0$ to $t_f = 1.1 \times 10^6$ years gives

$$\begin{aligned} d &= \int_0^{t_f} v(t)\mathrm{d}t = \int_0^{t_f} \frac{a't}{\sqrt{1 + (a't/c)^2}}\mathrm{d}t \\ &= \frac{c^2}{a'}\left[\sqrt{1 + (a't/c)^2} - 1\right] \approx 1 \times 10^{22}\text{ m}. \end{aligned} \tag{S.41}$$

The total distance is thus $d_{\text{tot}} \approx 2 \times 10^{22}$ m.

3. Check that during the journey the speed of light has never been surpassed.

Solution. Here we simply use Eq. (S.37). We clearly see that for $t \to \infty$ the speed $v \to c$.