Exercise 1. *Relativistic particle in a constant, uniform magnetic field*

Consider a point particle with mass m, charge q, initial velocity $\mathbf{v}_0$ and initial position $\mathbf{x}_0$, moving in a constant, uniform magnetic field $\mathbf{B} = B\,\hat{\mathbf{e}}_z$, parallel to the z axis. Let the 4-momentum be $p^\mu = (\frac{\mathcal{E}}{c}, \mathbf{p})$.

1. Show that the energy of the particle is constant in time, e.g.

$$\dot{\mathcal{E}} = 0.$$

2. Find the trajectory of the particle.

3. What are the differences between the classical and the relativistic trajectory?

Solution.

1. In general, we can always write that

$$\frac{d}{d\,t}(p^\mu p_\mu) = \frac{d}{d\,t}(m^2c^4) = 0.$$

Therefore

$$0 = p^\mu\,\frac{d}{d\,t}p_\mu = \frac{1}{c^2}\mathcal{E}\frac{d}{d\,t}\mathcal{E} - \mathbf{p}\cdot\frac{d}{d\,t}\mathbf{p}.$$

Knowing that $\dot{\mathbf{p}} = \mathbf{F}$ and that $\mathbf{v} = \frac{c^2\mathbf{p}}{\mathcal{E}}$, we get that

$$\dot{\mathcal{E}} = \mathbf{F}\cdot\mathbf{v}.$$

Since the magnetic force is orthogonal to $\mathbf{v}$, then $\dot{\mathcal{E}} = 0$.

2. The relativistic equations of motion

$$\frac{d}{dt}\mathbf{p} = \mathbf{F}, \tag{S.1}$$

in this case are just

$$\dot{\mathbf{p}} = q\,\mathbf{v}\times\mathbf{B},$$

with initial conditions

$$\mathbf{x}(t=0) = \mathbf{x_0} = (x_0, y_0, z_0), \qquad \mathbf{v}(t=0) = \mathbf{v}_0 = (v_{0,x}, v_{0,y}, v_{0,z}).$$

The equations of motion can be recast as

$$\frac{d\mathbf{v}}{d\,t} = \frac{c^2}{\mathcal{E}}\,q\,\mathbf{v}\times\mathbf{B}$$

since $\mathcal{E}$ is constant. In components, it becomes

$$\frac{d\,v_x}{d\,t} = \omega\,v_Y,\ \frac{d\,v_y}{d\,t} = -\omega\,v_x,\ \frac{d\,v_z}{d\,t} = 0,$$

where $\omega = \frac{qc^2B}{\mathcal{E}}$.
The equation for the z direction has the simple solution

$$z(t) = z_0 + v_{0,z}t.$$

For the transverse directions, we can rewrite the equations as

$$\frac{d}{d\,t}(v_x + iv_y) = -i\omega(v_x + iv_y),$$

whose solution is

$$v_x + iv_y = a\,\mathrm{e}^{-i\omega\, t}. \tag{S.2}$$

a is of course $a = v_{o,x} + iv_{o,y}$, but in order to get the trajectory, it's better to rewrite it as

$$a = |a|\mathrm{e}^{-i\alpha},$$

where $\alpha = \arctan\left(\frac{v_y}{v_x}\right)$. Then, we can integrate eq. (S.2) and get

$$x(t) + iy(y) = \frac{i\,|a|}{\omega}\,\mathrm{e}^{-i(\omega t+\alpha)} + q_0, \tag{S.3}$$

where q_0 is a complex integration constant. Then, we can get $x,\ y$ by taking Re, Im of the complex solution:

$$x(t) = \frac{|a|}{\omega}\sin(\omega t+\alpha) + \mathrm{Re}(q_0),$$
$$y(t) = \frac{|a|}{\omega}\cos(\omega t+\alpha) + \mathrm{Im}(q_0),$$

where $\mathrm{Re}(q_0)$ and $\mathrm{Im}(q_0)$ are the algebraic solutions to the equations $x(t=0) = x_0,\ \ y(t=0) = y_0$.

It is thus evident that the trajectory is a spiral whose axis is parallel to the direction of the magnetic field.

3. The classical trajectory is also a spiral with axis parallel to $\mathbf{B}$; however, the classical frequency is $\omega_{Cl} = \frac{q\,B}{m}$, which is (of course) the nonrelativistic limit of our ω (recall $\omega = \frac{q\,B}{\gamma m}$) .

Exercise 2. *Lorentz transformations for the Electromagnetic field*

a) Prove that under a general Lorentz transformation the $\vec{E}$ and $\vec{B}$ fields transform as follows:

$$\vec{E}' = \gamma\big(\vec{E} + c\,\vec{\beta}\times\vec{B}\big) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E}), \tag{1}$$

$$\vec{B}' = \gamma\big(\vec{B} - c^{-1}\,\vec{\beta}\times\vec{E}\big) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B}), \tag{2}$$

where $\vec{\beta} = \vec{v}/c$, $\gamma = (1-\beta^2)^{-1/2}$ and c is the speed of light.

Solution. We have

$$F'^{\mu\nu} = \Lambda^{\mu}_{\rho}\,\Lambda^{\nu}_{\sigma}\,F^{\rho\sigma}$$

where Λ is given by

$$\begin{bmatrix} \gamma & -\gamma\,\beta_x & -\gamma\,\beta_y & -\gamma\,\beta_z \\ -\gamma\,\beta_x & 1+(\gamma-1)\dfrac{\beta_x^2}{\beta^2} & (\gamma-1)\dfrac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\dfrac{\beta_x\beta_z}{\beta^2} \\ -\gamma\,\beta_y & (\gamma-1)\dfrac{\beta_y\beta_x}{\beta^2} & 1+(\gamma-1)\dfrac{\beta_y^2}{\beta^2} & (\gamma-1)\dfrac{\beta_y\beta_z}{\beta^2} \\ -\gamma\,\beta_z & (\gamma-1)\dfrac{\beta_z\beta_x}{\beta^2} & (\gamma-1)\dfrac{\beta_z\beta_y}{\beta^2} & 1+(\gamma-1)\dfrac{\beta_z^2}{\beta^2} \end{bmatrix}$$

In components:

$$
\begin{aligned}
F^{0i\,\prime} = -E^{i\prime} &= \Lambda^0_\rho\, \Lambda^i_\sigma\, F^{\rho\sigma} \\
&= \Lambda^0_0\, \Lambda^i_\sigma\, F^{0\sigma} + \Lambda^0_k\, \Lambda^i_\sigma\, F^{k\sigma} \\
&= \Lambda^0_0\, \Lambda^i_l\, F^{0l} + \Lambda^0_k\, \Lambda^i_0\, F^{k0} + \Lambda^0_k\, \Lambda^i_l\, F^{kl} \\
&= \Lambda^0_0 \left(\delta^i_l + (\gamma-1)\frac{\beta^i\beta^l}{\beta^2} \right) F^{0l} + \Lambda^i_0\,(-\gamma\,\vec\beta\cdot\vec E) + \Lambda^i_l\,(-\gamma\,\beta^k\, F^{kl}) \\
&= \gamma \left(F^{0i} + (\gamma-1)\frac{\beta_i}{\beta^2}(-\vec\beta\cdot\vec E) \right) - \gamma\,\beta^i\,(-\gamma\,\vec\beta\cdot\vec E) \\
&+ \left(\delta^i_l + (\gamma-1)\frac{\beta_i\beta_l}{\beta^2} \right) (-\gamma\,\beta^k\, F^{kl}) \\
&= -\gamma\, E^i + \frac{\gamma^3}{\gamma+1}\beta^i(-\vec\beta\cdot\vec E) + \gamma^2\,\beta^i\,\vec\beta\cdot\vec E \\
&+ \left(-\gamma\,\beta^k\, F^{ki} - \gamma\,(\gamma-1)\frac{\beta^i}{\beta^2}\,(\beta^l\beta^k\, F^{kl}) \right) \\
&= -\gamma\, E^i + \frac{\gamma^2}{\gamma+1}\beta^i\,\vec\beta\cdot\vec E + \gamma\,\beta^k\,\epsilon^{kil} B^l \\
&= -\gamma\, E^i + \frac{\gamma^2}{\gamma+1}\beta^i\,\vec\beta\cdot\vec E - \gamma\,(\vec\beta\times\vec B)^i
\end{aligned}
$$

where we use

$$\beta^2 = \frac{\gamma^2-1}{\gamma^2}$$

and the antisymmetry of F^{kl}. Then we have

$$\vec E' = \gamma\,\vec E - \frac{\gamma^2}{\gamma+1}\vec\beta\,(\vec\beta\cdot\vec E) + \gamma\,(\vec\beta\times\vec B) \tag{S.4}$$

For the B field

$$
\begin{aligned}
F^{ij\,\prime} = -\epsilon^{ijk} B^{k\prime} &= \Lambda^i_\rho\, \Lambda^j_\sigma\, F^{\rho\sigma} \\
&= \left(\Lambda^i_0\, \Lambda^j_k - \Lambda^i_k\, \Lambda^j_0 \right) F^{0k} + \Lambda^i_k\, \Lambda^j_l F^{kl} \\
&= \left[-\gamma\,\beta^i \left(\delta^j_k + (\gamma-1)\frac{\beta^j\beta^k}{\beta^2} \right) + \gamma\,\beta^j \left(\delta^i_k + (\gamma-1)\frac{\beta^i\beta^k}{\beta^2} \right) \right] F^{0k} \\
&+ \left(\delta^i_k + (\gamma-1)\frac{\beta^i\beta^k}{\beta^2} \right) \left(\delta^j_l + (\gamma-1)\frac{\beta^j\beta^l}{\beta^2} \right) F^{kl} \\
&= -\gamma \left[\beta^i\, F^{0j} - \beta^j\, F^{0i} + \frac{\gamma-1}{\beta^2} \left(\beta^i\,\beta^j\,\beta^k - \beta^i\,\beta^k\,\beta^j \right) F^{0k} \right] \\
&+ F^{ij} + \frac{\gamma-1}{\beta^2} \left(F^{il}\,\beta^j\,\beta^l + F^{kj}\,\beta^i\,\beta^k \right) + \frac{(\gamma-1)^2}{\beta^4}\beta^i\,\beta^j\,(\beta^k\,\beta^l\,F^{kl}) \\
&= \gamma\,\epsilon^{ijk}(\vec\beta\times\vec E)^k + F^{ij} - \frac{\gamma-1}{\beta^2}\epsilon^{ijk} \left[\beta^k(-\vec\beta\cdot\vec B) + \beta^2\, B^k \right] \\
&= \gamma\,\epsilon^{ijk}(\vec\beta\times\vec E)^k + \epsilon^{ijk} \left[\frac{\gamma-1}{\beta^2}\,\beta^k(\vec\beta\cdot\vec B) - \gamma\, B^k \right]
\end{aligned}
$$

and hence

$$\vec B' = \gamma(\vec B - \vec\beta\times\vec E) - \frac{\gamma^2}{\gamma+1}\vec\beta(\vec\beta\cdot\vec B) \tag{S.5}$$

b) Argue what happens to the angle between the $\vec E$ and $\vec B$ fields under a general boost transformation.

Solution. Since

$$\vec B\cdot\vec E$$

is Lorentz invariant, and the angle between the fields is given by

$$\cos\theta = \frac{\vec{B}\cdot\vec{E}}{|\vec{B}|\,|\vec{E}|}$$

then θ in general will change, unless, the fields are orthogonal in the original frame.

Exercise 3. *Electrodynamics in a Covariant formalism*

a) Given the electromagnetic field tensor $F^{\mu\nu}$ with components

$$F^{0i} = -E^i, \qquad F^{ij} = -\epsilon^{ijk} B_k, \qquad F^{\mu\nu} = -F^{\nu\mu}, \tag{3}$$

with $\epsilon_{123} = +1$, compute

$$-\frac{1}{4}\,F^{\mu\nu}\,F_{\mu\nu}, \qquad \epsilon_{\mu\nu\rho\sigma}\,F^{\mu\nu}\,F^{\rho\sigma} \tag{4}$$

in terms of the $\vec{E}$ and $\vec{B}$ fields.

Solution.

(a)

$$F^{\mu\nu}\,F_{\mu\nu} = F^{0\nu}\,F_{0\nu} + F^{1\nu}\,F_{1\nu} + F^{2\nu}\,F_{2\nu} + F^{3\nu}\,F_{3\nu} \tag{S.6}$$

$$= \underbrace{F^{00}\,F_{00}}_{0} + \Big(\underbrace{F^{01}\,F_{01} + \cdots + F^{03}\,F_{03}}_{-E^2}\Big) + \Big(\underbrace{F^{10}\,F_{10} + \cdots + F^{30}\,F_{30}}_{-E^2}\Big) \tag{S.7}$$

$$+ \Big(\underbrace{F^{11}\,F_{11} + \cdots + F^{13}\,F_{13}}_{B_z^2+B_y^2}\Big) + \Big(\underbrace{F^{21}\,F_{21} + \cdots + F^{23}\,F_{23}}_{B_z^2+B_x^2}\Big) + \Big(\underbrace{F^{31}\,F_{31} + \cdots + F^{33}\,F_{33}}_{B_y^2+B_x^2}\Big) \tag{S.8}$$

$$= -2\,E^2 + 2\,B^2 \tag{S.9}$$

(b) Let us first define a new tensor,

$$\left(\tilde{F}\right)_{\lambda\sigma} = \epsilon_{\mu\nu\lambda\sigma} F^{\mu\nu} \tag{S.10}$$

and calculate the components,

$$\left(\tilde{F}\right)_{01} = \epsilon_{\mu\nu\,01} F^{\mu\nu} = \epsilon_{2301} F^{23} + \epsilon_{3201} F^{32} = \epsilon_{0123} F^{23} - \epsilon_{0123} F^{32} \tag{S.11}$$

$$= \epsilon_{0123} F^{23} + \epsilon_{0123} F^{23} = 2\,F^{23} = -2\,B_x. \tag{S.12}$$

Analogous we find for the others non vanishing components

$$\left(\tilde{F}\right)_{02} = -2\,B_y \tag{S.13}$$

$$\left(\tilde{F}\right)_{03} = -2\,B_z \tag{S.14}$$

$$\left(\tilde{F}\right)_{12} = -2\,E_z \tag{S.15}$$

$$\left(\tilde{F}\right)_{13} = 2\,E_y \tag{S.16}$$

$$\left(\tilde{F}\right)_{23} = -2\,E_x \tag{S.17}$$

Computing now the full expression we get

$$\epsilon_{\mu\nu\lambda\sigma}\,F^{\mu\nu}\,F^{\lambda\sigma} = \left(\tilde{F}\right)_{\lambda\sigma} F^{\lambda\sigma} = 8\,\vec{B}\cdot\vec{E}. \tag{S.18}$$

b) Show that all the Maxwell equations

$$\partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0 \tag{5}$$
$$\vec{\nabla} \cdot \vec{B} = 0 \tag{6}$$

are equivalent to

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \tag{7}$$

Solution.

(a) For $\mu = 0, \nu = i, \lambda = k$ we have

$$\partial_0 F_{ik} + \partial_i F_{k0} + \partial_k F_{0i} = 0 \tag{S.19}$$
$$-\partial_0(\epsilon_{ikl} B^l) - \partial_i E_k + \partial_k E_i = 0 \tag{S.20}$$
$$\Rightarrow \partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0 \tag{S.21}$$

while for $\mu = j, \nu = i, \lambda = k$ we have

$$\partial_j F_{ik} + \partial_i F_{kj} + \partial_k F_{ji} = 0 \tag{S.22}$$
$$\partial_j B^j + \partial_i B^i + \partial_k B^k = 0 \tag{S.23}$$
$$\Rightarrow \vec{\nabla} \cdot \vec{B} = 0 \tag{S.24}$$

c) Given the Energy-momentum tensor

$$T^{\mu\nu}_{em} = F^\mu_\rho F^{\rho\nu} + \frac{1}{4}\, g^{\mu\nu}\, F_{\rho\sigma} F^{\rho\sigma} \tag{8}$$

compute the components T^{00}_{em}, T^{0i}_{em}, T^{ij}_{em} in terms of the $\vec{E}$ and $\vec{B}$ fields.

Solution.

$$T^{00}_{em} = F^0_\rho F^{\rho 0} + \frac{1}{4}\, g^{00}\, F_{\rho\sigma} F^{\rho\sigma} \tag{S.25}$$
$$= \vec{E}^2 + \frac{1}{4}\,(F_{0k} F^{0k} + F_{ik} F^{ik}) \tag{S.26}$$
$$= \vec{E}^2 + \frac{1}{2}\,(-\vec{E}^2 + \vec{B}^2) \tag{S.27}$$
$$= \frac{\vec{E}^2 + \vec{B}^2}{2} \tag{S.28}$$

and for T^{0i}_{em} we get

$$T^{0i} = F^0_\rho F^{\rho i} + \frac{1}{4} \underbrace{g^{0i}}_{0} F_{\rho\sigma} F^{\rho\sigma} \tag{S.29}$$
$$= g^{0\lambda} F_{\lambda\rho} F^{\rho i} = g^{00} F_{0\rho} F^{\rho i} = F_{00} F^{0i} + F_{01} F^{1i} + F_{02} F^{2i} + F_{03} F^{3i} \tag{S.30}$$
$$= E_x F^{1i} + E_y F^{2i} + E_z F^{3i} \tag{S.31}$$

which translates into

$$i = 1: \qquad -E_z B_y + E_y B_z \tag{S.32}$$
$$i = 2: \qquad -E_x B_z + E_z B_x \tag{S.33}$$
$$i = 3: \qquad -E_y B_x + E_x B_y \tag{S.34}$$

We then recognize the vector product,

$$T^{0i} = -\left(\vec{B} \times \vec{E}\right)^i. \tag{S.35}$$

We can write the tensor as,

$$T^{\mu\nu} = \begin{pmatrix} \frac{1}{2}(E^2+B^2) & S_x & S_y & S_z \\ S_x & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ S_y & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ S_z & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix} \tag{S.36}$$

where

$$\vec{S} = \vec{E} \times \vec{B} \tag{S.37}$$

is the Poyinting vector and

$$\sigma_{ij} = E_i E_j + B_i B_j - \frac{1}{2}(E^2 + B^2)\delta_{ij} \tag{S.38}$$

is the Maxwell stress tensor.
Notice that the metric we are using is $g^{\mu\nu} = diag(+,-,-,-)$.

d) Show that the Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$ is invariant under proper Lorentz transformations.

Solution. We first calculate the Lorentz transformation for ϵ^{0123}, e.g.

$$\left(\epsilon^{0123}\right)' = \Lambda^0_\alpha \Lambda^1_\beta \Lambda^2_\gamma \Lambda^3_\delta \epsilon^{\alpha\beta\gamma\delta} = det(\Lambda) = 1 = \epsilon^{0123} \tag{S.39}$$

Since $\epsilon^{\mu\nu\rho\sigma}$ is totally antisymmetric it follows,

$$\left(\epsilon^{\mu\mu\nu\rho}\right)' = \Lambda^\mu_\alpha \Lambda^\mu_\beta \Lambda^\nu_\gamma \Lambda^\rho_\delta \epsilon^{\alpha\beta\gamma\delta} = 0 = \epsilon^{\mu\mu\nu\rho} \tag{S.40}$$

and the same for all the other cases in which two indeces are the same. In all the other cases we get -1 which proof that the Levi-civita tensor is Lorentz invariant, i.e.

$$\epsilon^{\mu\nu\rho\sigma} = \Lambda^\mu_\alpha \Lambda^\nu_\beta \Lambda^\rho_\gamma \Lambda^\sigma_\delta \epsilon^{\alpha\beta\gamma\delta} \tag{S.41}$$