Exercise 1. ***Getting familiar with four-vectors***

In this exercise, you will become more familiar with the four-vector manipulations.
The greek indices μ, ν, ... take values $0,\ 1,\ ...,\ d$, where d is the dimension of space (we are used to it being equal to 3, but it can be kept general).

1. **Derivative of a position vector:** Let now $x^\mu = (x^0,\ x^1,\ ...,\ x^d)$ and $\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial x^0},\ \frac{\partial}{\partial x^1},\ ...)$. What is $\partial_\mu x^\mu$? Can you see that it is indeed a Lorentz scalar?

   Solution. The results is $1 + 1 + ... + 1 = d + 1$.

2. **Lorentz tensors:** A general tensor can be written as an object with multiple indices, both up and down, i.e. $A^{\mu\nu\rho...}_{\gamma\delta\sigma...}$. Its transformation properties follow from the transformation properties of the tensor product of vectors, i.e. $x'^\mu y'^\nu = \Lambda^\mu_\sigma \Lambda^\nu_\gamma x^\sigma y^\gamma$ implies that $A'^{\mu\nu} = \Lambda^\mu_\sigma \Lambda^\nu_\gamma A^{\sigma\gamma}$.
   Prove however, that not every tensor can be written as a product of vectors, that is e.g. argue that it is not always possible to find $a^\mu,\ b^\nu$ such, that $S^{\mu\nu} = a^\mu b^\nu$ (even if $S^{\mu\nu}$ is symmetric).

   Solution. For example, a general 2-index tensor has d^2 components. On the other hand, a tensor product of 2 vectors only has $2d$ components.

3. **Symmetric and antisymmetric tensors**: In the following, let $A^{\mu\nu}$ be an antisymmetric tensor, that is $A^{\mu\nu} = -A^{\nu\mu}$ and $S^{\mu\nu}$ be a symmetric tensor: $S^{\mu\nu} = S^{\nu\mu}$.

   (a) Show that the (anti)symmetry of the tensor is preserved by the Lorentz transformations.

   Solution. As always, $A'^{\mu\nu} = \Lambda^\mu_\rho \Lambda^\nu_\sigma A^{\rho\sigma}$. Then

   $$A'^{\nu\mu} = \Lambda^\nu_\rho \Lambda^\mu_\sigma A^{\rho\sigma} = -\Lambda^\nu_\rho \Lambda^\mu_\sigma A^{\sigma\rho} \tag{S.1}$$

   Then exchange the dummy indices ρ and σ, thus getting:

   $$A'^{\nu\mu} = -\Lambda^\nu_\sigma \Lambda^\mu_\rho A^{\rho\sigma} = -\Lambda^\mu_\rho \Lambda^\nu_\sigma A^{\rho\sigma} = -A'^{\mu\nu} \tag{S.2}$$

   Similarly for $S^{\mu\nu}$.

   (b) Prove that $A^{\mu\nu} S_{\mu\nu} = 0$.

   Solution.

   $$A^{\mu\nu} S_{\mu\nu} = -A^{\nu\mu} S_{\nu\mu} = -A^{\mu\nu} S_{\mu\nu} \tag{S.3}$$

   thus proving the result.

Let us now introduce a concept of symmetrization and antisymmetrization of a 2-index tensor [1]. For an arbitrary tensor $C^{\mu\nu}$ introduce it's symmetrisation: $C^{(\mu\nu)} = \frac{1}{2}(C^{\mu\nu} + C^{\nu\mu})$ and antisymmetrisation: $C^{[\mu\nu]} = \frac{1}{2}(C^{\mu\nu} - C^{\nu\mu})$.

[1] This concept can be generalised to higher order tensors. We will however not need it here

(c) Show that a general rank-2 tensor can be uniquely decomposed into the symmetric and antisymmetric part: $C^{\mu\nu} = C^{(\mu\nu)} + C^{[\mu\nu]}$.

Solution. Just plug in the definitions of symmetrisation and antisymmetrisation.

Exercise 2. *(More) General Lorentz transformation*

One usually performs Lorentz boosts along a specific axis, say x or z. In this exercise, we will perform a more general one, with a velocity vector in a specific plane.
In the lab frame, you are moving with a velocity $\vec{v} = (v, 0, 0)$, that is with a velocity v along the x axis. Another object is moving with the velocity $\vec{u} = (u_x, u_y, 0)$ as measured in the same frame.
What should your velocity v be so that the velocity of the object along the y axis, as measured by you, is also u_y.
(NB: $v = 0$ is clearly an answer; you are supposed to find the non-trivial one).
Hint: Recall that the four-velocity of an object in its rest frame is $(c, \vec{0})$.

Solution. The four-velocity of the *another object* in its rest frame is $(1, 0, 0, 0)$ (using $c = 1$). First perform a Lorentz transformation with the velocity $-\vec{u}$ (thus getting it's form in the lab frame), and then again a Lorentz boost with a velocity $\vec{v}$. This is the four-velocity of the *object* as seen in the observer's rest frame. It's form is $u_{obs} = (\gamma, \gamma\beta_x, \gamma\beta_y, 0)$. Specifically:

$$u_{obs} = \begin{pmatrix} \frac{1-u_x v}{\sqrt{1-u_x^2-u_y^2}\sqrt{1-v^2}} \\ \frac{u_x - v}{\sqrt{1-u_x^2-u_y^2}\sqrt{1-v^2}} \\ \frac{u_y}{\sqrt{1-u_x^2-u_y^2}} \\ 0 \end{pmatrix} \tag{S.4}$$

From the last component one can easily read-off $\beta_{obs,y}$ (equal to $u_{obs,y}$ in the case of $c = 1$):

$$u_{obs,y} = \frac{u_y\sqrt{1-v^2}}{1-u_x v}. \tag{S.5}$$

And we want to solve the equation:

$$u_{obs,y} = u_y \tag{S.6}$$

for the variable v. This has 2 solutions: $v = 0$ and $v = \frac{2u_x}{1+u_x^2}$.
NB, a general Lorentz transformation has the form:

$$\begin{bmatrix} \gamma & -\gamma\,\beta_x & -\gamma\,\beta_y & -\gamma\,\beta_z \\ -\gamma\,\beta_x & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\gamma\,\beta_y & (\gamma-1)\frac{\beta_y\beta_x}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\gamma\,\beta_z & (\gamma-1)\frac{\beta_z\beta_x}{\beta^2} & (\gamma-1)\frac{\beta_z\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{bmatrix}$$

Thus to obtain the u_{obs} one needs to indeed multiply the matrix with $\vec{\beta} = \vec{v} = (v,\ 0,\ 0)$, the one with $\vec{\beta} = -\vec{u} = (-u_x,\ -u_y,\ 0)$ and the vector $(1,\ 0,\ 0,\ 0)$.

Exercise 3. ***Relativistic Doppler effect***

1. In 1929 Edwin Hubble proposed a law that was later named after him - that the galaxies seem to drift away from us the faster, the further away they already are:

$$v_{rec} = Hd \tag{1}$$

where v_{rec} is the Galaxy's receding velocity, d - it's distance from us and H is the Hubble's constant.
But how does one measure the velocity of the galaxy? One of the possibilities is to use the shift in the frequency of the emission lines of the elements constituting the matter of the galaxy. As hydrogen is the most common element, of special use is the *blue line of hydrogen* with the frequency of $\lambda = 434nm$ as measured in the laboratory on Earth (that is with the hydrogen atom at rest with respect to the observer).

   (a) Assume that in an astrophysical experiment a physicist detects a line of $\lambda' = 600nm$ in the spectrum of a distant galaxy. Assuming that it indeed corresponds to the blue line of hydrogen calculate the velocity with which the galaxy receds from the Earth.

   Solution. From the transverse Doppler effect formula we have that:

$$\nu' = \nu\sqrt{\frac{1-u/c}{1+u/c}}, \tag{S.7}$$

   where $\nu = \frac{c}{\lambda}$ and $\nu' = \frac{c}{\lambda'}$. Solving this for u we get $u = 0.31c$

   (b) The latest measure of the Hubble constant by the Planck experiment gave a result of $67.8\frac{km}{s\ Mpc}$ [2]. For dimensional reasons, the inverse of the Hubble constant, *the Hubble time*, can be used as a rough estimate of the age of the Universe. See how good it is by comparing it with the most recent estimate of 13.7 billion years.

   Solution. $\frac{1}{H} \approx 14 \times 10^9$ years, so it is reasonably close. That suggests that the expansion rate of the Universe has been nearly constant over it's history.

2. Consider now a frame $\mathcal{O}'$, moving away from the Earth (system $\mathcal{O}$), in which a light source is at rest. In this frame the light is emitted at an angle θ' with respect to the x'-axis and its frequency is ν'.

   (a) Assuming the Earth as fixed and the system $\mathcal{O}'$ moving with constant velocity v along the x-axis (i.e. x- and x'-axes are parallel, see figure below), find the observed angle θ from the x-axis in the system $\mathcal{O}$.

   Solution. In the system $\mathcal{O}'$, according to the above description of the problem, the photon has the following energy and momentum

$$E' = h\nu'$$
$$p_x' = \frac{h\nu'}{c}\cos\theta'$$
$$p_y' = \frac{h\nu'}{c}\sin\theta'$$
$$p_z' = 0.$$

[2] Mpc - megaparsec. $1pc \approx 3.086 \times 10^{13}km$

Changing now to the Earth reference frame $\mathcal{O}$ gives

$$\begin{aligned}
E &= \gamma\left(h\nu' + v\frac{h\nu'}{c}\cos\theta'\right) \overset{!}{=} h\nu \\
p_x &= \gamma\left(\frac{h\nu'}{c}\cos\theta' + v\frac{h\nu'}{c^2}\right) \overset{!}{=} \frac{h\nu}{c}\cos\theta \\
p_y &= \frac{h\nu'}{c}\sin\theta' \overset{!}{=} \frac{h\nu}{c}\sin\theta \\
p_z &= 0.
\end{aligned} \tag{S.8}$$

Equations (S.8) leads to

$$\nu\cos\theta = \gamma\nu'\left(\cos\theta' + v/c\right) \tag{S.9}$$

$$\nu\sin\theta = \nu'\sin\theta' \tag{S.10}$$

$$\nu = \gamma\nu'\left(1 + \frac{v}{c}\cos\theta'\right). \tag{S.11}$$

Dividing Eq. (S.9) by Eq. (S.11) we get the desired relation for the angle θ in system $\mathcal{O}$

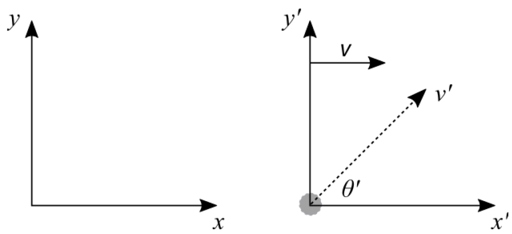
$$\cos\theta = \frac{\cos\theta' + v/c}{1 + (v/c)\cos\theta'}. \tag{S.12}$$

(b) Assuming $v = 0.7c$, for which angle θ' does the Doppler shift vanish? What do you expect in the non-relativistic case ($v \ll c$)?

Solution. Here we just use Eq. (S.11) and demand that $\nu = \nu'$. This leads to

$$\theta' = \arccos\left[\frac{v}{c}\left(\gamma^{-1} - 1\right)\right]. \tag{S.13}$$

For $v = 0.7c$ we get $\theta' \approx 101.5°$. For the non-relativistic limit ($v \ll c$) we expect to have vanishing Doppler shift for $\theta' = 90°$.



Exercise 4. *Relativistic force*

Consider a constant force $F = const.$ acting on a free particle.

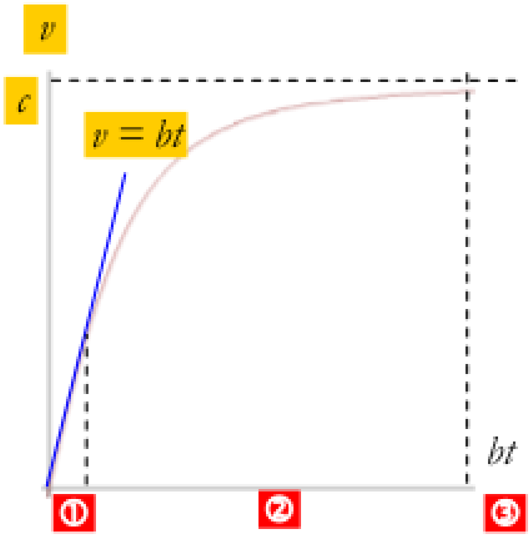
1. Give the expression of the velocity of the particle as a function of time and consider the limit $t \to \infty$.

Solution. From the relation $F = \dot{p}$, and assuming the motion starts from rest at the origin at time $t = 0$, we find that

$$p = Ft. \tag{S.14}$$

We also know that

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}. \tag{S.15}$$



Solving for v gives

$$v = \frac{(F/m)t}{\sqrt{1+\left(\frac{Ft}{mc}\right)^2}} \tag{S.16}$$

from which we easily see that $v \to c$ for $t \to \infty$

2. Give the expression of the position as a function of time.

Solution. We just integrate the above expression (S.16):

$$x = \frac{F}{m}\int_0^t \frac{t'}{\sqrt{1+\left(\frac{Ft'}{mc}\right)^2}}\mathrm{d}t' = \frac{mc^2}{F}\sqrt{1+\left(\frac{Ft'}{mc}\right)^2}\Bigg|_0^t = \frac{mc^2}{F}\left[\sqrt{1+\left(\frac{Ft}{mc}\right)^2}-1\right]. \tag{S.17}$$

Here, instead of the usual non-relativistic parabola $x = 1/2\,(F/m)t^2$, we get an hyperbola given by Eq. (S.17).