Exercise 1. *d'Alembert operator and reference frames*

The d'Alembert operator is defined as

$$\Box = \frac{1}{c^2}\frac{\partial}{\partial t^2} - \vec{\nabla}^2. \tag{1}$$

Is this operator invariant under a change of reference frame? For this, check how the operator transforms under coordinate transformations. Consider relative motion in x direction under

a) Galilean transformations

b) Special Relativity

What is the significance of this in the context of Maxwell's equations?

Solution.

a) Let us consider a reference frame moving with velocity v in x direction relative to the initial reference frame. In a Galilean coordinate transformation, we then use

$$x' = x - vt \tag{S.1}$$
$$y' = y \tag{S.2}$$
$$z' = z \tag{S.3}$$
$$t' = t. \tag{S.4}$$

The corresponding derivatives are

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} = \frac{\partial}{\partial x} \tag{S.5}$$
$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} \tag{S.6}$$
$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z} \tag{S.7}$$
$$\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}. \tag{S.8}$$

Also the second derivatives with respect to y', z' and x' stay unchanged, For t' we get

$$\frac{\partial^2}{\partial t'^2} = \frac{\partial^2}{\partial t^2} + v^2\frac{\partial^2}{\partial x^2} + 2v\frac{\partial^2}{\partial x \partial t}. \tag{S.9}$$

The new d'Alembert operator then becomes

$$\Box' = \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial y'^2} - \frac{\partial^2}{\partial z'^2} \tag{S.10}$$
$$= \Box + \frac{v^2}{c^2}\frac{\partial^2}{\partial x^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial x \partial t}, \tag{S.11}$$

and is hence not invariant under a Galilean coordinate transform. This is worrisome since it would render Maxwell's equations frame dependent — this has been a key observation for the development of special relativity.

b) In Special Relativity, the transformation is

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \tag{S.12}$$

$$y' = y \tag{S.13}$$

$$z' = z \tag{S.14}$$

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}. \tag{S.15}$$

For the derivatives this translates to

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} = \frac{1}{\sqrt{1 - v^2/c^2}}\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right) \tag{S.16}$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} \tag{S.17}$$

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z} \tag{S.18}$$

$$\frac{\partial}{\partial t} = \frac{1}{\sqrt{1 - v^2/c^2}}\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right). \tag{S.19}$$

The (non-trivial) second derivatives are thus

$$\frac{\partial^2}{\partial x^2} = \frac{1}{1 - v^2/c^2}\left(\frac{\partial^2}{\partial x'^2} + \frac{v^2}{c^4}\frac{\partial^2}{\partial t'^2} - \frac{2v}{c^2}\frac{\partial^2}{\partial x'\partial t'}\right) \tag{S.20}$$

and

$$\frac{\partial^2}{\partial t^2} = \frac{1}{1 - v^2/c^2}\left(\frac{\partial^2}{\partial t'^2} + v^2\frac{\partial^2}{\partial x'^2} - 2v\frac{\partial^2}{\partial x'\partial t'}\right), \tag{S.21}$$

and so

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \frac{1}{1 - v^2/c^2}\left(\frac{1}{c^2}\frac{\partial^2}{\partial t'^2} + \frac{v^2}{c^2}\frac{\partial^2}{\partial x'^2} - \frac{2v}{c^2}\frac{\partial^2}{\partial x'\partial t'} - \frac{\partial^2}{\partial x'^2} - \frac{v^2}{c^2}\frac{\partial^2}{c^2\partial t'^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial x'\partial t'}\right) \tag{S.22}$$

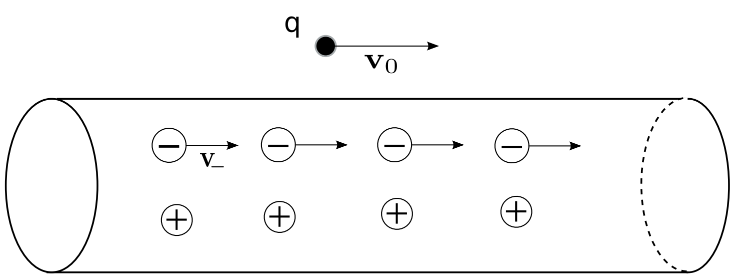
$$= \frac{1 - v^2/c^2}{1 - v^2/c^2}\left(\frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial x'^2}\right) \tag{S.23}$$

$$= \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial x'^2} \tag{S.24}$$

is left invariant. This implies that also $\Box$ is left invariant under Lorentz transformations, and so Maxwell's equations are frame-independent.

Exercise 2. *Electric and magnetic fields in different frames*

Consider a wire and let the laboratory frame $\mathcal{S}$ be the one where the wire is at rest, with a charged particle (with charge q) moving with constant speed $\mathbf{v}_0$ parallel to the wire at a distance r, as in the figure below. The radius of the wire is small in comparison with r.



Consider a current flowing through the wire as the flow of negative charges having all the same velocity $\mathbf{v}_-$. Moreover, let both the positive and negative charge densities ρ_+, ρ_- be constant and uniform inside the wire, and let the wire be globally neutral in the frame $\mathcal{S}$.

a) Evaluate the force $\mathbf{F}$ acting on the charged particle in the laboratory frame $\mathcal{S}$.

b) Evaluate the charge densities ρ_+, ρ_- in the rest frame $\mathcal{S}'$ of the particle. Show that in this frame the net charge density is nonzero.

Hints:

- If you consider a constant and uniform charge density ρ in its rest frame, then the charge density seen by an observer moving with speed $\mathbf{u}$ with respect to the charges is

$$\rho'|_{\text{boost}} = \frac{\rho|_{\text{rest}}}{\sqrt{1-\frac{u^2}{c^2}}};$$

- Remember the proper formula for adding velocities in Special relativity.

c) Evaluate the force $\mathbf{F}'$ acting on the particle in the frame $\mathcal{S}'$ and show that

$$|\mathbf{F}'| = \frac{|\mathbf{F}|}{\sqrt{1-\frac{v_0^2}{c^2}}}.$$

d) Now consider another frame $\mathcal{S}''$ which moves at a velocity $v_{\mathcal{S}''}$ with respect to the laboratory frame $\mathcal{S}$. What is the magnitude of the force, $|\mathbf{F}''|$ acting on the charged particle in this frame?

Hints: Now the charged particle is not at rest, and also the net charge in the wire is not zero. Therefore both electric and magnetic fields contribute to the total electromagnetic force. So you need to rewrite the magnetic field that you found in part a in this boosted frame. Take also into account the fact that the positive charges move in this frame. To find the force due to the electric field recycle your results from parts b and c by just replacing the velocity of the frame $\mathcal{S}'$ by the velocity of the frame $\mathcal{S}''$.

Solution.

a) In general, the total electromagnetic force on the particle is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. In the frame $\mathcal{S}$, the net charge of the wire is zero by assumption, so $\mathbf{E} = 0$. The magnetic field of a wire has been derived in the lecture and is given by

$$\mathbf{B} = \frac{1}{4\pi\epsilon_0 c^2}\frac{2I}{r}\mathbf{e}_\phi\,,$$

at a distance r from the wire and with ϕ being the cylindrical coordinate around the wire. As also seen in the lecture, the current I produced by the motion of the negative charges is $I = S\rho_- v_-$, where S is the cross-sectional area of the wire, which we will ignore from this point on. This means that we will effectively write $I = \rho_- v_-$ with ρ_- a $1D$ density.
The force on the particle will thus point in the radial direction ($\mathbf{v}_0 = v_0\mathbf{e}_z$) and its magnitude is

$$|\mathbf{F}| = \frac{q}{2\pi\epsilon_0}\frac{\rho_-}{r}\frac{v_0 v_-}{c^2}\,.$$

b) Due to the Lorentz-contraction of lengths (and Lorentz-invariance of electric charge), the charge densities transform as given in the hint when going from a density's rest frame to a boosted frame, as it has also been derived in the lecture:

$$\rho'|_{\text{boost}} = \frac{\rho|_{\text{rest}}}{\sqrt{1-\frac{u^2}{c^2}}};$$

Thus, the density of the positive charges (whose rest frame is $\mathcal{S}$) in the frame $\mathcal{S}'$ is given by

$$\rho'_+ = \frac{\rho_+}{\sqrt{1-\frac{v_0^2}{c^2}}}\,.$$

The case of the negative charges is more difficult, as neither $\mathcal{S}$ nor $\mathcal{S}'$ is their rest frame. So we will have to boost them *first* into their rest frame, and *then* into the particle's rest frame. The first step will incorporate a boost with the velocity v_-. For the second step, we need to use the proper addition of velocities in Special Relativity, and find that the particle's velocity in the rest frame of the negative charges is given by

$$v_0' = \frac{v_0 - v_-}{1 - \frac{v_0 v_-}{c^2}} \,.$$

Adding up and using the fact that the net charge in the frame $\mathcal{S}$ is zero ($\rho_- = -\rho_+$), the total charge density in the frame $\mathcal{S}'$ is found to be

$$\rho' = \rho_+' + \rho_-' = -\rho_- \left(\frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}} - \frac{\sqrt{1 - \frac{v_-^2}{c^2}}}{\sqrt{1 - \frac{1}{c^2}\left(\frac{v_0 - v_-}{1 - \frac{v_0 v_-}{c^2}}\right)^2}} \right) .$$

This is in general nonzero, and we observe that we obtain the result from the lecture by setting $v_- = v_0$.

c) In the frame $\mathcal{S}'$, the magnetic force is zero because the particle is at rest. The electric field produced by a wire with charge density ρ' is pointing out radially, and its magnitude is given by

$$E' = \frac{1}{2\pi\epsilon_0}\frac{\rho'}{r} \quad \Rightarrow \quad |\mathbf{F}'| = \frac{q}{2\pi\epsilon_0}\frac{|\rho'|}{r}$$

We therefore find the desired relation between the forces $|\mathbf{F}|$ and $|\mathbf{F}'|$ if

$$|\rho'| = \frac{v_0 v_- \rho_-}{c^2} \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}} \,,$$

and indeed (with $\frac{v_0}{c} \equiv a$, $\frac{v_-}{c} \equiv b$),

$$|\rho'| = \rho_- \left(\frac{1}{\sqrt{1-a^2}} - \frac{\sqrt{1-b^2}}{\sqrt{1 - \left(\frac{b-a}{1-ab}\right)^2}} \right) = \frac{\rho_-}{\sqrt{1-a^2}} \cdot \mathcal{J} ,$$

where

$$\begin{aligned} \mathcal{J} =& 1 - \sqrt{\frac{(1-a^2)(1-b^2)}{1 - \left(\frac{b-a}{1-ab}\right)^2}} = 1 - \sqrt{\frac{(1-a^2)(1-b^2)}{(1-ba)^2 - (b-a)^2}(1-ba)^2} \\ =& 1 - \sqrt{\frac{1 - a^2 - b^2 + a^2b^2}{1 + a^2b^2 - 2ab + 2ab - a^2 - b^2}(1-ab)^2} = ab \,. \end{aligned}$$

d) First we look at the force due to magnetic field: $\mathbf{F} = q\,(\mathbf{v} \times \mathbf{B})$. In the frame $\mathcal{S}''$ the particle's velocity is:

$$v_0'' = \frac{v_0 - v_{S''}}{1 - \frac{v_0 v_{S''}}{c^2}} \,.$$

The magnitude of the magnetic field is:

$$|\mathbf{B}''| = \frac{1}{4\pi\epsilon_0 c^2}\frac{2I''}{r} \,,$$

Now the positive charges also move with a velocity $-v_{S''}$ in this frame, so the current is:

$$I'' = -\rho_+'' v_{S''} + \rho_-'' v_-''$$

Now we need to find what v_-'', ρ_+'' and ρ_-'' are.

The velocity of the negative charges in frame $\mathcal{S}''$ is:

$$v_-'' = \frac{v_- - v_{S''}}{1 - \frac{v_- v_{S''}}{c^2}} \,.$$

The positive charge density in frame $\mathcal{S}''$ is:

$$\rho''_+ = \frac{\rho_+}{\sqrt{1-\frac{v''^2_S}{c^2}}} = \frac{-\rho_-}{\sqrt{1-\frac{v''^2_S}{c^2}}}.$$

The negative charge density is:

$$\rho''_- = \rho_- \frac{\sqrt{1-\frac{v_-^2}{c^2}}}{\sqrt{1-\frac{1}{c^2}\left(\frac{v_{S''}-v_-}{1-\frac{v_{S''}v_-}{c^2}}\right)^2}}$$

Combining everything we find for the current:

$$I'' = -\rho''_+ v_{S''} + \rho''_- v''_- \tag{S.25}$$

$$= \left(\frac{\rho_-}{\sqrt{1-\frac{v''^2_S}{c^2}}}\right) v_{S''} + \rho_- \left(\frac{\sqrt{1-\frac{v_-^2}{c^2}}}{\sqrt{1-\frac{1}{c^2}\left(\frac{v_{S''}-v_-}{1-\frac{v_{S''}v_-}{c^2}}\right)^2}}\right)\left(\frac{v_- - v_{S''}}{1-\frac{v_- v_{S''}}{c^2}}\right) \tag{S.26}$$

$$= \frac{\rho_- v_-}{\sqrt{1-\frac{v''^2_S}{c^2}}} \tag{S.27}$$

$$= \frac{I}{\sqrt{1-\frac{v''^2_S}{c^2}}} \tag{S.28}$$

And the force due to magnetic field is found from plugging in the expressions for I'' and v''_0 readily:

$$|\mathbf{F}''_1| = q v''_0 |\mathbf{B}''| = \frac{v''_0 q}{4\pi\epsilon_0 c^2}\frac{2I''}{r} \tag{S.29}$$

$$= \frac{q}{2\pi\epsilon_0 c^2}\left(\frac{v_0 - v_{S''}}{1-\frac{v_0 v_{S''}}{c^2}}\right)\left(\frac{\rho_- v_-}{\sqrt{1-\frac{v''^2_S}{c^2}}}\right)\frac{1}{r} \tag{S.30}$$

Now we look at the force due to electric field. We repeat the steps in part c:

$$E'' = \frac{1}{2\pi\epsilon_0}\frac{\rho''}{r} \quad\Rightarrow\quad |\mathbf{F}''| = \frac{q}{2\pi\epsilon_0}\frac{|\rho''|}{r}$$

where $|\rho''|$ is the net charge in frame $\mathcal{S}''$: $\rho'' = \rho''_+ + \rho''_-$. Again using the results from part c, but with the replacements $a = v_{S''}/c$ and $b = v_-/c$ we find:

$$|\rho''| = \rho''_+ + \rho''_- = \frac{\rho_-}{\sqrt{1-\frac{v^2_{S''}}{c^2}}}\left(\frac{v_{S''}v_-}{c^2}\right)$$

Then the force due to the electric field becomes:

$$|\mathbf{F}''_2| = \frac{q}{2\pi\epsilon_0}\frac{\rho_-}{\sqrt{1-\frac{v^2_{S''}}{c^2}}}\left(\frac{v_{S''}v_-}{c^2}\right)\frac{1}{r}$$

Combining $|\mathbf{F}''_1|$ and $|\mathbf{F}''_2|$ we get:

$$|\mathbf{F}''_1| + |\mathbf{F}''_2| = \frac{q}{2\pi\epsilon_0}\frac{1}{rc^2}\left(\frac{\rho_- v_-}{\sqrt{1-\frac{v^2_{S''}}{c^2}}}\right)\left[\left(\frac{v_0 - v_{S''}}{1-\frac{v_0 v_{S''}}{c^2}}\right) + v_{S''}\right] \tag{S.31}$$

Exercise 3. *Green's functions*

The Green's function physically represents a response of the system if a unit point source (charge) is applied to the system. Mathematically, the Green's function is the kernel of an integral operator that represents an inverse of the differential operator. (In what follows, we do not mean to be rigorous, we just want to give an idea of the use of Green's functions).

Consider the following problem:

$$Au(\vec{x}) = f(\vec{x}) \tag{2}$$
$$B_1 u = \vec{a} \tag{3}$$
$$B_2 u = \vec{b} \tag{4}$$

Where A is a differential operator (e.g. Laplacian), and B_1 and B_2 are boundary value operators (compare the discretised version of this with Sheet 3, question 3). We can write the differential operator A together with boundary conditions B, as a differential operator L, s.t:

$$Lu(x) = f(x) \tag{5}$$

1. When does the inverse of the L exist?

   Solution. This inverse exists if $Lu = 0$ has only a trivial solution.

2. If the inverse exists, what form does it take? Can you recognise the Green's function?

   Solution.
   $$u(x) = (L^{-1})(x) = \int_V G(x, y) f(y) dy^3, \tag{S.32}$$
   with $G(x, y)$ a Green's function. G is a kernel of the integral transform defining u.

3. If f represents a unit point source acting at the point y, what does the Green's function represent?

   Solution. G represents a solution to Eq.5 - it is a response to a unit, point source at y.

4. Compare the discretised version of taking the inverse of operator L, with the matrix inversion in solving the matrix equations (e.g. solving Poisson equation numerically in Sheet 3, Question 3). What is the important difference between the problem in Eq.5 and the matrix equations?

   Solution. We can write the matrix problem as $A\vec{u} = \vec{f}$, with A a regular $n \times n$ matrix, and $\vec{u}$ and $\vec{f}$ n-vectors. Then the solution is:
   $$u_i = \sum_{j=1}^{n} A_{ij}^{-1} f_j \tag{S.33}$$
   This is a discrete analog to:
   $$u(x) = (L^{-1})(x) = \int_V G(x, y) f(y) dy^3 \tag{S.34}$$
   and the Green's function seems to be the analogue of the inverse matrix A^{-1}. But actually the inverse of the matrix is analogue to the function G together with the boundary conditions. Hence the difference to keep in mind, is that when solving differential equations and using a Green's function, a solution is very often not possible without taking into account the boundary conditions. Remember that in the Sheet 3, after discretising Poisson equation and obtaining the matrix equation $A\vec{\phi} = \vec{\rho}$, to get the correct solution we needed to replace the RHS with $\vec{b}$, which included the boundary conditions on the potential ϕ.

Exercise 4. ***Getting more familiar with magnetostatics***

In this exercise, we would like to understand a few more things about magnetic field and vector potential. To this end, we will first consider a sheet with current and then ask for more general considerations in the context of boundary conditions in magnetostatics.

a) Imagine a sheet with surface current $\vec{K}$. What can you say about the magnetic field above and below the surface? How does this compare to the electric field in the presence of surface charges?

b) Take a sphere of radius R. Suppose that we know the vector potential on the surface is given by $\vec{A}_S$, how can we calculate the potential outside the sphere? How did we do it in electrostatics? Give an outline only!

Solution.

a) Imagine a sheet with surface current $\vec{K}$. Since

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{S.35}$$

we know from Gauss' law that for the fields right above and below the surface

$$0 = \int_V d^3x \vec{\nabla} \cdot \vec{B} = \hat{n} \cdot (\vec{B}_2 - \vec{B}_1) \tag{S.36}$$

and hence the normal component of the magnetic field is continuous at the surface.

Similarly, using Ampere's law we see that the tangential field is discontinuous:

$$\hat{n} \times (\vec{B}_2 - \vec{B}_1) = \vec{K}. \tag{S.37}$$

b) If we know the value of A at the surface, we can proceed analogously to the electrostatic case, since we have

$$\vec{A}(\vec{r}) = \frac{1}{4\pi\epsilon_0 c^2} \int d^3\vec{r}' \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|}. \tag{S.38}$$

Hence we just pick the corresponding Greens function (that vanishes on the surface) and calculate component-wise:

$$A_i(\vec{r}) = \frac{1}{4\pi\epsilon_0 c^2} \int_V d^3\vec{r}' J_i(\vec{r}') G(\vec{r}, \vec{r}') - \frac{1}{4\pi} \oint_S A_i(\vec{r}') \frac{\partial G}{\partial r} da'. \tag{S.39}$$

The solution will hence be analogous to ex. 2 of sheet 4.