Exercise 1. *Spherical Hamonics*

In this exercise we want to become more confident with the Spherical Harmonics.

1. Starting from

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos(\theta))\,e^{im\phi} \tag{1}$$

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}\,(1-x^2)^{m/2}\,\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \tag{2}$$

with $x=\cos(\theta)$, derive the expressions for

$$Y_{00},\, Y_{1,m},\, Y_{2,m}, \qquad m\in[-l,l] \tag{3}$$

Solution. We have

$$Y_{00} = \sqrt{\frac{1}{4\pi}} \tag{S.1}$$

$$Y_{1,-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin(\theta)\,e^{-i\,\phi} \tag{S.2}$$

$$Y_{1,0} = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos(\theta) \tag{S.3}$$

$$Y_{1,1} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin(\theta)\,e^{i\,\phi} \tag{S.4}$$

$$Y_{2,-2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2(\theta)\,e^{-2i\,\phi} \tag{S.5}$$

$$Y_{2,-1} = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin(\theta)\,\cos(\theta)\,e^{-i\,\phi} \tag{S.6}$$

$$Y_{2,0} = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2(\theta)-1) \tag{S.7}$$

$$Y_{2,1} = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin(\theta)\,\cos(\theta)\,e^{i\,\phi} \tag{S.8}$$

$$Y_{2,2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2(\theta)\,e^{2i\,\phi} \tag{S.9}$$

(S.10)

2. Draw the following functions

$$|Y_{00}|^2,\, |Y_{1,m}|^2,\, |Y_{2,m}|^2$$

for $m\in[-l,l]$ in a 3D plot (using for example Mathematica).

Solution. For instance, you can plot $|Y_{1,1}|^2$ with

SphericalPlot3D[(*Sqrt*[3/(8 ∗ *Pi*)] ∗ *Sin*[*theta*])^2, {*theta*, 0, *Pi*}, {*phi*, 0, 2 ∗ *Pi*},
PlotRange− > *All*, *PlotPoints*− > 50]

3. Verify the orthogonality conditions explicitly for the previous functions.
Notice that $Y_{l,m}(\theta,\phi)$ with m odd (even) are always odd (even) in θ.

Solution. The normalization and orthogonality conditions are

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin(\theta) d\theta \, Y^*_{l' m'}(\theta, \phi) \, Y_{lm}(\theta, \phi) = \delta_{l\, l'} \delta_{m\, m'} \tag{S.11}$$

Using the Spherical Harmonics found in (S.1) these conditions are immediately verified.

Exercise 2. *Spherical cavity and spherical functions*

Consider a sphere of radius a where the surface of the upper hemisphere has a potential $+\Phi_0$ and the surface of the lower hemisphere has a potential $-\Phi_0$, that is:

$$\Phi_0(\theta', \phi') = \begin{cases} +\Phi_0 & \text{for } \theta' \in [0, \frac{\pi}{2}] \\ -\Phi_0 & \text{for } \theta' \in (\frac{\pi}{2}, \pi]. \end{cases} \tag{4}$$

As you know from the lecture (method of image), in this case the Green Function is given by

$$G(\vec{r}, \vec{r}\,') = \frac{1}{|\vec{r} - \vec{r}\,'|} - \frac{a}{r' \left| \vec{r} - \frac{a^2}{r'^2} \vec{r}\,' \right|} \tag{5}$$

where $\vec{r}\,'$ refers to a unit source outside the sphere and $\vec{r}$ to the point where the potential is evaluated.

1. Using the expansion

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta', \phi') \, Y_{lm}(\theta, \phi) \tag{6}$$

where $r_<(r_>)$ is the smaller (larger) of $|\vec{r}|$ and $|\vec{r}\,'|$, show that the Green Function (5) can be written as

$$G(\vec{r}, \vec{r}\,') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a} \left(\frac{a^2}{r\, r'} \right)^{l+1} \right] Y^*_{lm}(\theta', \phi') \, Y_{lm}(\theta, \phi) \tag{7}$$

Solution. Just use (6) for both terms of the Green function.

2. Using Dirichlet boundary conditions, show that the potential outside the sphere has following the expansion

$$\Phi(r, \theta, \phi) = \sum_{l,m} \left(\frac{a}{r} \right)^{l+1} Y_{lm}(\theta, \phi) \int \Phi_0(\theta', \phi') Y^*_{lm}(\theta', \phi') d\Omega'. \tag{8}$$

which tends to 0 as $r \to \infty$.

Solution. The general solution to the Poisson equation for the potential with Dirichlet boundary conditions is

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{r}\,') \, G(\vec{r}, \vec{r}\,') \, d^3r' - \frac{1}{4\pi} \oint_S \Phi(\vec{r}\,') \left(-\frac{\partial G}{\partial r'} \right) da' \tag{S.12}$$

The first term vanishes since there is no charge density outside the sphere; for the second term, we have in this case that $r_< = r'$ and $r_> = r$, so that taking the derivative

$$\begin{aligned}\left.\frac{\partial G}{\partial r'}\right|_{r'=a} &= 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left.\left(\frac{a^{2l+1}}{r'^{\,l+2}}(l+1) + l\, r'^{\,l-1}\right)\right|_{r'=a} \frac{1}{r^{l+1}}\, Y^*_{lm}(\theta', \phi')\, Y_{lm}(\theta, \phi) \\ &= 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{a^{l-1}}{r^{l+1}} (2l+1)\, Y^*_{lm}(\theta', \phi')\, Y_{lm}(\theta, \phi) \qquad \text{(S.13)}\end{aligned}$$

and integrating with $da' = a^2\, d\Omega$ gives the result.

3. Now calculate the potential outside the sphere using (8) up to the terms of order a^4.

   Hint. Notice that only terms with odd l will survive in the expansions (8) for the potential given by (4).

   Solution. We first notice that

   $$P_l(-x) = \frac{1}{2^l l!} \frac{d^l}{d(-x)^l} \left((-x)^2 - 1\right)^l = (-1)^l P_l(x) \qquad \text{(S.14)}$$

   Since now the potential (4) is an odd function of $x = \cos\theta$ and we integrate over $x \in [-1, 1]$, the integrand has to be even in order for the integral not to vanish. Therefore only odd P_l's survive.

   Then, to the order required we only need to integrate the potential with P_1 and P_3. We will use $x = \cos\theta$ as the variable. Since according to the discussion above the integrand is even, we can just integrate over $x \in [0, 1]$ and then multiply the result by 2. Thus for the term of the expansions (8) proportional to $P_1(\cos\theta)$ we get:

   $$2\,\Phi_0 \times 2\pi \times \frac{3}{4\pi} \left(\int_0^1 x dx\right) P_1(\cos\theta) = \frac{3\,\Phi_0}{2} P_1(\cos\theta) \qquad \text{(S.15)}$$

   where Φ_0 is the potential on the sphere, 2 was discussed above, 2π is a result of the integration over ϕ, $\frac{3}{4\pi}$ comes from the normalization of the spherical harmonics and eventually x is just $P_1(x)$. Analogously, for the second term we get:

   $$2\,\Phi_0 \times 2\pi \times \frac{7}{4\pi} \left(\int_0^1 \frac{1}{2}(5x^3 - 3x) dx\right) P_3(\cos\theta) = -\frac{7\,\Phi_0}{8} P_3(\cos\theta) \qquad \text{(S.16)}$$

   as $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.
   Plugging the above expressions into the expansions (8) we finally obtain:

   $$\Phi(r, \theta, \phi) = \frac{3}{2}\,\Phi_0 \left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8}\,\Phi_0 \left(\frac{a}{r}\right)^4 P_3(\cos\theta) + O(r^{-6}) \qquad \text{(S.17)}$$

Exercise 3. *Surface charge density*

As in the previous exercise, consider a spherical shell of radius R. The sphere has a charge density $\sigma(\theta)$ such that the surface potential has the following form

$$\Phi(R, \theta) = V_0 + V_1 \cos\theta + V_2 \cos 2\theta, \qquad (9)$$

where V_0, V_1, V_2 are constants and θ is the polar angle.

1. Find the potential $\Phi(r, \theta)$ both inside and outside the spherical shell. Can be useful, in this case, to rewrite the potential in a unique form as

   $$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left(a_l r^l + b_l r^{-(l+1)}\right) P_l(\cos\theta), \qquad (10)$$

where P_l are the Legendre polynomials.

Hint. Use that $\Phi \to 0$ for $r \to \infty$ and the orthogonality of the Legendre polynomials.

Solution. We use the relation $\cos 2\theta = 2\cos^2\theta - 1$ to rewrite the surface potential (9) as

$$\Phi(R, x) = V_0 + V_1 x + V_2(2x^2 - 1), \tag{S.18}$$

where we set $x = \cos\theta$.

In order to find the coefficients a_l and b_l we start by multiplying Eq. (10) by $P_k(x)$. Then we integrate it with respect to x and we make use of the orthogonality relation for Legendre polynomials, obtaining

$$a_k r^k + b_k r^{-(k+1)} = \frac{2k+1}{2}\int_{-1}^{1} \Phi(r, x) P_k(x) \mathrm{d}x. \tag{S.19}$$

Inside the sphere we require the potential to be regular, therefore we have that $b_k = 0 \; \forall \; k$. Then, in order to find the coefficients a_k we use the knowledge of the potential at $r = R$ by plugging Eq. (S.18) into (S.19). We obtain

$$a_0 = \frac{1}{3}(3V_0 - V_2), \tag{S.20}$$

$$a_1 = \frac{V_1}{R}, \tag{S.21}$$

$$a_2 = \frac{4V_2}{3R^2}. \tag{S.22}$$

For $k > 2$ we have $a_k = 0$. We thus find the following potential inside the spherical shell

$$\Phi_{\text{in}}(r, \theta) = \frac{1}{3}(3V_0 - V_2) + V_1 \cos\theta \frac{r}{R} + \frac{2V_2}{3}(3\cos^2\theta - 1)\frac{r^2}{R^2}. \tag{S.23}$$

Similarly, using the fact that $\Phi \to 0$ for $r \to \infty$, we find the following potential outside the sphere

$$\Phi_{\text{out}}(r, \theta) = \frac{1}{3}(3V_0 - V_2)\frac{R}{r} + V_1 \cos\theta \frac{R^2}{r^2} + \frac{2V_2}{3}(3\cos^2\theta - 1)\frac{R^3}{r^3}. \tag{S.24}$$

2. Calculate the electric field $E(r, \theta)$.

Solution. The electric field is given by $\mathbf{E} = -\nabla\Phi$. Due to the symmetry of the problem we use here the gradient in spherical coordinates, which is given by

$$\nabla = \frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\mathbf{e}_\varphi. \tag{S.25}$$

A simple evaluation leads to the following expression for the electric field inside the spherical shell

$$E_{\text{in}}(r, \theta) = \left(\frac{V_1}{R} + 4V_2\cos\theta\frac{r}{R^2}\right)\sin\theta\,\mathbf{e}_\theta - \left(\frac{V_1}{R}\cos\theta + \frac{4V_2}{3}(3\cos^2\theta - 1)\frac{r}{R^2}\right)\mathbf{e}_r. \tag{S.26}$$

In the same way we find the electric field outside

$$E_{\text{out}}(r, \theta) = \left(V_1\frac{R^2}{r^3} + 4V_2\cos\theta\frac{R^3}{r^4}\right)\sin\theta\,\mathbf{e}_\theta + \left(\frac{1}{3}(3V_0 - V_2)\frac{R}{r^2} + 2V_1\cos\theta\frac{R^2}{r^3} + 2V_2(3\cos^2\theta - 1)\frac{R^3}{r^4}\right)\mathbf{e}_r. \tag{S.27}$$

3. Find the surface charge density $\sigma(\theta)$.

Hint. Use the fact that the component of the electric field orthogonal to the spherical surface undergoes a jump at $r = R$.

Solution. The discontinuity of the perpendicular component of the electric field across the shell boundaries is related to the surface charge density in the following way

$$\mathbf{E}_{\text{out}}^{\perp} - \mathbf{E}_{\text{in}}^{\perp} = \frac{\sigma}{\epsilon_0}. \tag{S.28}$$

Using the two equations (S.26) and (S.27) and the last expression we find the following surface charge density

$$\sigma(\theta) = \frac{\epsilon_0}{R}\left(V_0 + \frac{4}{3}V_2 + 3V_1\cos\theta + 5V_2\cos 2\theta\right). \tag{S.29}$$

Exercise 4. *Conductors and capacities*

In this problem we introduce and analyze the concept of capacity constants for arrays of conductors. Inside a conductor the electric field $\mathbf{E}$ vanishes and the electric potential is constant for static (i.e. equilibrium) situations. We consider finitely many perfect conductors described by spatially separated sets $A_1, \ldots, A_r$ for some $r \in \mathbb{N}$. Assume that they are carrying total charges $Q_1, \ldots, Q_r$ and that there exists a (unique) equilibrium charge density ρ (which of course vanishes outside the conductors).

1. The potential $V_i = V_i(\{Q_k\})$ is the potential of the i−th conductor. Show that

$$V_i(\{\lambda Q_k\}) = \lambda V_i(\{Q_k\}) \tag{11}$$

for any $\lambda \in \mathbb{R}$, using the explicit integral expression for a potential generated by a given charge distribution.

Solution. Let's denote the equilibrium charge density for the new situation also by the symbol ρ. Since all conductors are perfect, their potentials are constant and we denote them by $V_1, \ldots, V_r$. They can explicitly be calculated by

$$V_i(\{Q_k\}) \equiv V_i(\rho) = \frac{1}{4\pi\varepsilon_0}\int_{\mathbb{R}^3} \mathrm{d}^3y\rho(\mathbf{y})\,\frac{1}{|\mathbf{x}_i - \mathbf{y}|} \tag{S.30}$$

where $\mathbf{x}_i$ is some point in A_i (which point in A_i one chooses is irrelevant, since the potential is constant over A_i) and ρ is determined by $\{Q_k\}$ and $\{A_k\}$. If we scale the total charges $\{Q_k\}$ with a factor λ, i.e. the conductors carry charges $\{\lambda Q_k\}$ then the corresponding equilibrium distribution is given by $\lambda\rho$. This follows directly from the *uniqueness of solution* and

$$\mathbf{E}[\rho](\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathbb{R}^3} \mathrm{d}^3y\rho(\mathbf{y})\,\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}, \tag{S.31}$$

which states that $\lambda\rho$ also generates a total electric field that vanishes in all A_i.

The linearity dependence of the potential on ρ and the linear dependence of the equilibrium distribution ρ on $\{Q_k\}$ leads to

$$V_i(\{\lambda Q_k\}) = \lambda V_i(\{Q_k\}) \qquad \forall\lambda. \tag{S.32}$$

2. Using Eq. 11 show that $V_i = V_i(\{Q_k\})$ depends linearly on $Q_1, \ldots, Q_r$, i.e.

$$V_i(\{Q_k\}) = \sum_{j=1}^{r} D_{ij}\,Q_j, \tag{12}$$

where $D_{ik} := \frac{\partial V_i}{\partial Q_k}$ depends only on $\{A_k\}$.

Solution. Taking the derivative of both sides of Eq. 11 with respect to λ yields

$$V_i(\{Q_k\}) = \sum_{j=1}^{r} \frac{\partial V_i}{\partial Q_j}(\{\lambda Q_k\})\, Q_j. \tag{S.33}$$

Since this holds for all λ it holds in particular for $\lambda = 0$, which finally leads to

$$V_i(\{Q_k\}) = \sum_{j=1}^{r} \frac{\partial V_i}{\partial Q_j}(\mathbf{0})\, Q_j \tag{S.34}$$

and by defining

$$D_{ij} := \frac{\partial V_i}{\partial Q_j}(\mathbf{0}) \tag{S.35}$$

we find

$$V_i(\{Q_k\}) = \sum_{j=1}^{r} D_{ij}\, Q_j, \tag{S.36}$$

where D_{ij} are independent of $\{Q_k\}$.

It turns out that $D = (D_{ik})$ is regular. We define $C = (C_{ik}) := D^{-1}$. Its components C_{ik} are called capacity constants and depend only on the geometry $\{A_k\}$.

3. Show that the total energy W of the equilibrium charge distribution $\rho \equiv \rho(\{Q_k\}, \{A_k\})$ in the situation of the previous problem can be expressed as

$$W = \frac{1}{2} \sum_{i,j=1}^{r} C_{ij}\, V_i\, V_j. \tag{13}$$

Solution. The equilibrium charge distribution ρ has the total energy

$$\begin{aligned}
W &= \frac{1}{2} \int_{\mathbb{R}^3} \mathrm{d}^3x \rho(\mathbf{x})\, V(\mathbf{x}) \\
&= \frac{1}{2} \sum_{i=1}^{r} \int_{A_i} \mathrm{d}^3x \rho(\mathbf{x})\, V(\mathbf{x}) \\
&= \frac{1}{2} \sum_{i=1}^{r} V_i \int_{A_i} \mathrm{d}^3x \rho(\mathbf{x}) \\
&= \frac{1}{2} \sum_{i=1}^{r} V_i\, Q_i \\
&= \frac{1}{2} \sum_{i,j=1}^{r} C_{ij}\, V_i\, V_j,
\end{aligned} \tag{S.37}$$

where in the second line we used that ρ is only supported in $A_1, \ldots, A_r$ and in the third line that the potential is constant in A_i.