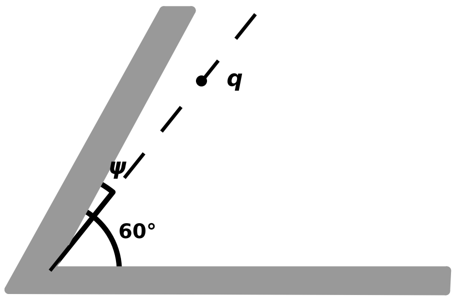
Exercise 1. *Method of Images*

Consider a point charge q located at any point between two grounded conductive metal plates that have $60°$ angle between them.



1. Calculate the electrostatic potential in the space between the plates using the method of images. In the figure sketch the needed mirror charges to determine the potential. Discuss how the result depends on the angle ψ.

   Solution. From Figure 1, we need 6 image charges to calculate the potential. The positions of the 6 charges in cylindrical coordinates are:

$$\begin{array}{ll} q & (r, \varphi, z_0) \\ q_1 = -q & \left(r, -\varphi + \frac{2\pi}{3}, z_0\right) \\ q_2 = q & \left(r, \varphi + \frac{2\pi}{3}, z_0\right) \\ q_3 = -q & \left(r, -\varphi - \frac{2\pi}{3}, z_0\right) \\ q_4 = q & \left(r, \varphi - \frac{2\pi}{3}, z_0\right) \\ q_5 = -q & (r, -\varphi, z_0) \end{array}$$

   The potential is

$$\phi(\vec{x}) = \frac{q}{4\pi\epsilon_0} \sum_{i=0}^{5} \frac{(-1)^i}{\sqrt{(x - r\cos\varphi_i)^2 + (y - r\sin\varphi_i)^2 + (z - z_0)^2}},$$

   where the index 0 is used for the point charge itself .

2. Determine the direction and the magnitude of the force $\vec{F}$ acting on the charge q.

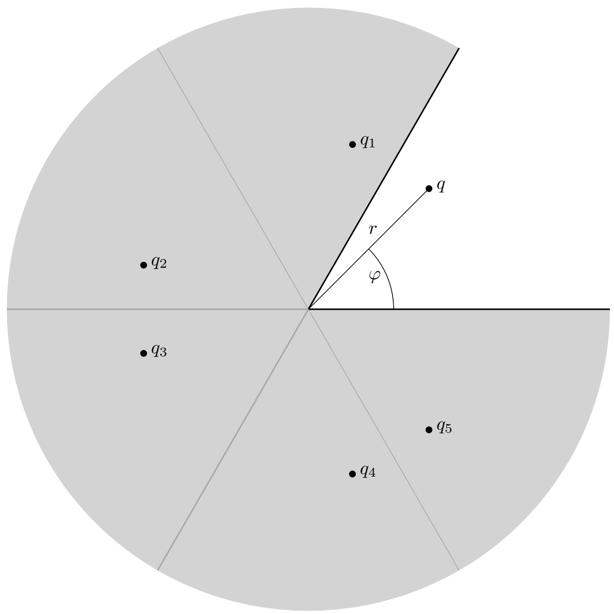


Figure 1: Position of the mirror charges.

Solution. The force acting on the charge q (at position $\vec{x}$) is

$$\begin{aligned}
F(\vec{x}) &= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{5}\frac{q_i}{\sqrt{(x-r\cos\varphi_i)^2+(y-r\sin\varphi_i)^2+(z_0-z_0)^2}^3}\begin{pmatrix} x-r\cos\varphi_i \\ y-r\sin\varphi_i \\ z_0-z_0\end{pmatrix} \\
&= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{5}\frac{q_i}{\sqrt{(r\cos\varphi-r\cos\varphi_i)^2+(r\sin\varphi-r\sin\varphi_i)^2}^3}\begin{pmatrix} r\cos\varphi-r\cos\varphi_i \\ r\sin\varphi-r\sin\varphi_i \\ 0\end{pmatrix} \\
&= \frac{q}{4\pi\epsilon_0}\sum_{i=1}^{5}\frac{(-1)^i}{\sqrt{2r^2(1-\cos(\varphi-\varphi_i))}^3}\begin{pmatrix} r\cos\varphi-r\cos\varphi_i \\ r\sin\varphi-r\sin\varphi_i \\ 0\end{pmatrix}
\end{aligned}$$

Exercise 2. *Solution of the Poisson Equation with Green's Function*

The general solution for the potential is given by:

$$\Phi(\vec{y}) = \frac{1}{4\pi\epsilon_0}\int_V G(\vec{x},\vec{y})\,\rho(\vec{x})\,d^3\vec{x} - \frac{1}{4\pi}\oint_S d\vec{S}\cdot\left(\Phi_s(\vec{x})\,\nabla_{\vec{x}}G(\vec{x},\vec{y}) - G(\vec{x},\vec{y})\,\nabla_{\vec{x}}\Phi_s(\vec{x})\right) \qquad (1)$$

where Φ_s is the surface potential.

Now consider the figure below, where the $z < 0$ region is filled with a conductor and the $z > 0$ region is free. We choose the Green's function such that it is zero on the surface of the conductor: $G = 0$ at $z = 0$. Therefore in Equation (1) the last term vanishes.

1. Now consider that the surface potential is not constant but proportional to $1/r$:

$$\Phi_s = \frac{V}{r}$$

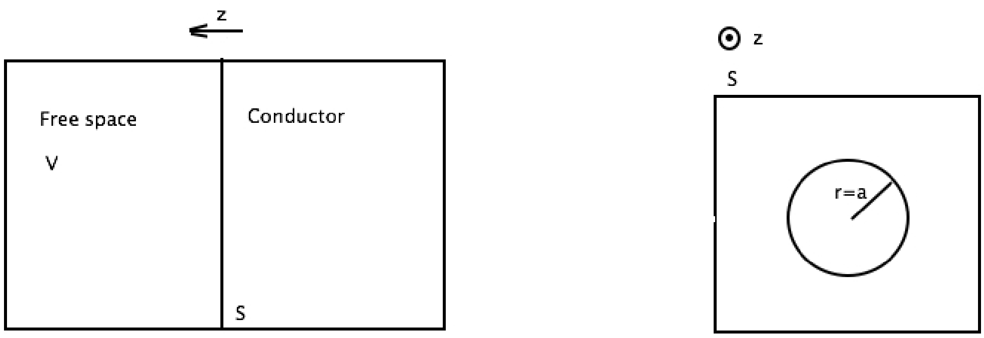


Figure 2: The conductor cross section and the surface of the conductor

where $r = \sqrt{x_1^2 + x_2^2}$ and V is a constant. Calculate the potential at a point on the z axis in the free part, i.e. $x_1 = x_2 = 0$. For this use the following Green's function:

$$G(\vec{x}, \vec{y}) = \left(\frac{1}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}} - \frac{1}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 + y_3)^2}} \right) \tag{2}$$

Note that now x_3 points in the direction of the z axis. Also there are no charges $\rho(\vec{x}) = 0$ in the region we are interested so the first term in Equation (1) vanishes. Use the cylindrical coordinates.

Solution. Since x_3 is in the direction of z, we make the following replacements for convenience:

$$x_1 \to x, \quad x_2 \to y, \quad x_3 \to z$$
$$y_1 \to x', \quad y_2 \to y', \quad y_3 \to z'$$

The Green's function can be calculated using the following conditions:

- $$\Delta^2 G = -\delta(x - x')\delta(y - y')\delta(z - z')$$
- Boundary condition: $G = 0$ when $z = 0$

The solution is (solving for the above function and adding the component for the mirror charge in the region $z < 0$):

$$G = \frac{1}{((x - x')^2 + (y - y')^2 + (z - z')^2)^{1/2}} - \frac{1}{((x - x')^2 + (y - y')^2 + (z + z')^2)^{1/2}}$$

We can calculate the potential in the region $z > 0$ by using the formula:

$$\Phi(\mathbf{z}) = \frac{1}{4\pi\epsilon_0} \int_V G\,\rho(\mathbf{z}')\,dV' - \frac{1}{4\pi} \oint_S \left(\Phi_s \frac{\partial G}{\partial n} - G \frac{\partial \Phi_s}{\partial n} \right) dS \tag{S.1}$$

We know that $z > 0$ is a free region, hence the first term above vanishes as $\rho = 0$. Also the last term above vanishes since G is zero on the surface. In the region $z > 0$, with $z' = 0$, we have:

$$\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial z'} = -2z \frac{1}{((x - x')^2 + (y - y')^2 + z^2)^{3/2}} \tag{S.2}$$

Then the potential becomes:

$$\Phi(\vec{z}) = \frac{z}{2\pi}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \frac{V}{(x'^2+y'^2)^{1/2}} \frac{1}{((x-x')^2+(y-y')^2+z^2)^{3/2}} \tag{S.3}$$

where we took the surface potential to be $\Phi_s = V/r' = V/(x'^2+y'^2)^{1/2}$. Changing to cylindrical coordinates and for $x = y = 0$:

$$\Phi(\vec{z}) = \frac{z}{2\pi}\int_0^{2\pi} d\theta \int_0^{\infty} dr'\, r' \frac{1}{r'} \frac{V}{(r'^2+z^2)^{3/2}} \tag{S.4}$$

$$= V\,\frac{1}{z} \tag{S.5}$$

2. Now assume that the surface of the conductor has the following potential:

$$\Phi_s = 0, r > a$$

$$\Phi_s = V, r < a$$

with $r = \sqrt{x_1^2 + x_2^2}$, as seen in Figure 2. Hence the potential is everywhere zero on the $z = 0$ plane, except for the circle of the radius a around the coordinate beginning. Use the Green's function to calculate the potential on the z-axis.

Solution. Now we have the following potential in this region:

$$\Phi(z) = \frac{z}{2\pi}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \frac{\Phi_s(x',y')}{((x-x')^2+(y-y')^2+z^2)^{3/2}} \tag{S.6}$$

With the Φ_s as given in the question, on the z-axis ($x = y = 0$)we have:

$$\Phi(z) = \frac{zV}{2\pi}\int_0^a \frac{2\pi r' dr'}{(r'^2+z^2)^{3/2})}$$

with $r'^2 = x'^2 + y'^2$. Hence, for the potential on the z-axis, one gets:

$$\Phi(z) = V\,(1 - \frac{z}{\sqrt{z^2+a^2}})$$

for $z > 0$.

3. Now imagine that the region $z > 0$ is not free, but includes a finite line charge of length L, as described in Sheet1, Question 3.2. This line charge is parallel to the surface of the conductor, and located at a distance b from it. Assume that the conductor is grounded so the surface potential is zero. How would you now calculate the potential in the arbitrary point in the region $z > 0$?

Solution. If the region $z > 0$ is not free, but has a line charge, we have that the first integral in the expression for the potential does not vanish. Let our line be placed at $y = 0$, at the distance $z = b$ from the surface and stretch from $x = -\frac{L}{2}$ to $x = \frac{L}{2}$. Let $\lambda = \frac{Q}{L}$ be the linear charge density. The (spatial) charge density is thus given by:

$$\rho(x,y,z) = \lambda\delta(z-b)\delta(y) \tag{S.7}$$

for $x \in [-\frac{L}{2}, \frac{L}{2}]$ and 0 everywhere else. We are therefore left with the integral:

$$\Phi(x,y,z) = \frac{\lambda}{4\pi\epsilon_0}\int_{-\frac{L}{2}}^{\frac{L}{2}} G(x,y,z;x',0,b)dx' \tag{S.8}$$

Using

$$\int \frac{1}{\sqrt{1+x^2}}dx = \text{arsinh}(x) = log(x+\sqrt{1+x^2}) \tag{S.9}$$

it integrates to:

$$\Phi(x,y,z) = \frac{\lambda}{4\pi\epsilon_0}\left(log\frac{L-2x+\sqrt{4(b-z)^2+(L-2x)^2+4y^2}}{-L-2x+\sqrt{(L+2x)^2+4y^2+4(b-z)^2}} - log\frac{L-2x+\sqrt{(L-2x)^2+4y^2+4(b+z)^2}}{-L-2x+\sqrt{(L+2x)^2+4y^2+4(b+z)^2}}\right). \tag{S.10}$$

One could also obtain this result with a *mirror line* placed on the other side of the grounded conducting surface.

Exercise 3. *Numerical solution of the Poisson equation*

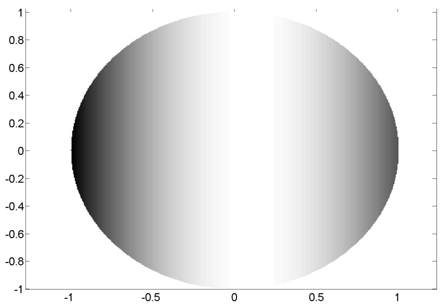
In this exercise we will see two simple examples of how electrostatic problems can be solved numerically.

a) Consider a disk of radius 1 centered at (0,0) with dimensionless charge density given by $\rho(x,y) = (x-0.1)^2$ as shown in the figure above.

Matlab pdetool provides an intuitive interface to set up a configuration, solve differential equations and visualize the results. Using Matlab pdetool calculate the potential with the Poisson equation

$$\nabla^2\Phi = -\rho$$

at any point of the disk, plot the equipotential lines and the electric field.
Assume Dirichlet boundary conditions (potential is zero at the domain boundary).
Note: Matlab can be found at all ETH publicly available computers. To start pdetool type pdetool in the Matlab command line. Choose Electrostatics mode. Use simple dimensionless units.



b) Solve the one-dimensional Poisson equation

$$\frac{\partial^2\Phi}{\partial x^2} = -\rho(x) \tag{3}$$

without using the Matlab PDE toolbox.

i) Write eq. 3 in a discretized form, using finite differences with a spatial separation Δx. Φ is then only defined on discrete points $\Phi(x_n)$ $(n = 1, 2, \cdots, N)$

ii) Set the Dirichlet boundary conditions: $\Phi(x_0) = c_0$ and $\Phi(x_N) = c_N$ with two constants c_0 and c_N.

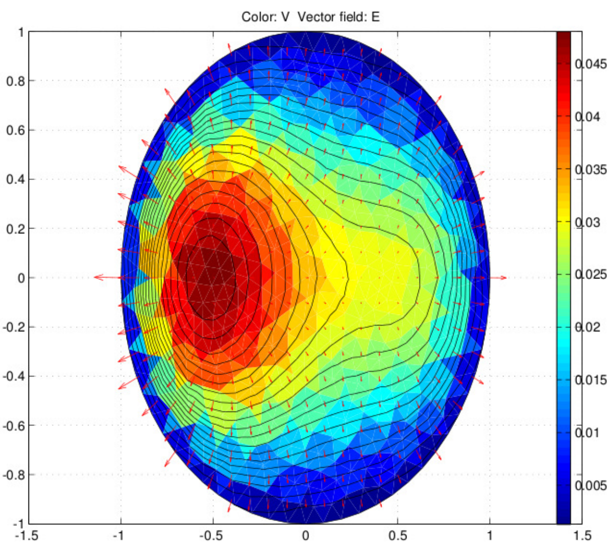
iii) The discretized Poisson is a set of (N-1) coupled linear equations and can be written in matrix form as $A\mathbf{\Phi} = \mathbf{b}$. Write out this system.

iv) Set the initial conditions $\rho(x) = (x - 0.1)^2$ and solve the system on a computer, with B.C. $c_0 = c_N = 0$.

v) Compare the result to the analytical solution.

Solution.

a) Use graphical interface to set up the disk configuration. Differential equation to be solved and boundary condition can be defined using the buttons PDE and $\partial\Omega$, respectively. Solution can be obtained using the "=" button.
To plot the equipotential lines and the electric field choose Plot→Parameters... and mark "Contour" and "Arrows" checkboxes.
The visualized solution should look like



b) 1) The first derivative using asymmetric finite difference is

$$\frac{\partial \Phi(x_n)}{\partial x} = \frac{\Phi(x_{n+1}) - \Phi(x_n)}{\Delta x} + \mathcal{O}(\Delta x) \quad \text{(S.11)}$$

$$= \frac{\Phi(x_n) - \Phi(x_{n-1})}{\Delta x} + \mathcal{O}(\Delta x) \quad \text{(S.12)}$$

The second derivative is a combination of S.11 and S.12:

$$\frac{\partial^2 \Phi(x_n)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\Phi(x_{n+1}) - \Phi(x_n)}{\Delta x} + \mathcal{O}(\Delta x)\right) \tag{S.13}$$

$$= \frac{1}{\Delta x}\left[\frac{\Phi(x_{n+1}) - \Phi(x_n)}{\Delta x} - \frac{\Phi(x_n) - \Phi(x_{n-1})}{\Delta x}\right] + \mathcal{O}((\Delta x)^2) \tag{S.14}$$

$$= \frac{\Phi(x_{n+1}) + \Phi(x_{n-1}) - 2\Phi(x_n)}{(\Delta x)^2} + \mathcal{O}((\Delta x)^2) \tag{S.15}$$

The discretized Poisson equation is then

$$\Phi_{n+1} + \Phi_{n-1} - 2\Phi_n = -(\Delta x)^2 \rho_n$$

2) $\Phi(x_0) = c_0$ and $\Phi(x_n) = c_n$

3) The system $A\mathbf{\Phi} = \mathbf{b}$ can be written as

$$\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \vdots \\ \Phi_{N-2} \\ \Phi_{N-1} \end{pmatrix} = -(\Delta x)^2 \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \vdots \\ \rho_{N-2} \\ \rho_{N-1} \end{pmatrix} - \begin{pmatrix} c_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ c_N \end{pmatrix}.$$

4) This system can be solved directly with Matlab using the following script:

Listing 1: solveFDm.m

```
close all;
clear all;

% input and mesh
sol = inline('-(1/12).*(x-0.1).^4 + (1/24)*(0.9^4-1.1^4).*x +(1/24)*(1.1^4+0.9^4)','x');
f   = inline('(x-0.1).^2','x');
M = 100;
h = 2/M;
x = linspace(-1,1,M+1)';

% compute finite difference matrix A and right hand side
e = ones(M-1,1);
A = spdiags([e, -2*e, e], -1:1, M-1, M-1);
rho = h^2*f(x(2:end-1));

% incorporate Dirichlet boundary data
phi = zeros(M+1,1);

% solve the linear system
phi(2:end-1) = - A\rho;

% plot solution
hold on
plot(linspace(-1,1,1000),sol(linspace(-1,1,1000)),'r-');
plot(x,phi,'bo:')
legend('exact','finite difference','Location','SouthWest')
xlabel('x')
ylabel('\phi')
```

Without using Matlab one can use the Jacobi relaxation method to solve the System. This is implemented in the following c++ code.

Listing 2: jacobi.cpp

```
#include <stdio.h>
#include <stdlib.h>
```

```cpp
#include <cmath>
#include <iostream>
#define L 100 //lenght of system
#define xmin -1.0
#define xmax 1.0

using namespace std;

double* lat = new double[L];
double* lat2 = new double[L];
double* rho = new double[L];
double delta = (xmax - xmin)/L;


int main(){
  double x = 0;
  bool flag = true;
  for(int i = 0; i < L; ++i){
    x = i*delta + xmin;
    rho[i] = -(x-0.1)*(x-0.1);   //set initial conditions
  }


  while(flag == true){
  flag = false;
  for(int i = 1; i < L-1; ++i){
        lat2[i] = 0.5 * (lat[i-1] + lat[i+1] - (delta*delta) * rho[i]);
        if(abs(lat2[i] - lat[i]) > 0.00000001){
          flag = true;
        }
  }
  for(int i = 1; i < L-1; ++i){
      lat[i] = lat2[i];
  }
}

  for(int i = 1; i < L-1; ++i){
     x = i*delta + xmin;
     cout << x << " " << lat[i] << endl;
  }

 return 0;
}
```