Exercise 1. ***Equilibrium in electrostatic field and behaviour inside a cavity of a conductor***

1. Can a point charge be in a stable mechanical equilibrium in the electric field of other charges?

   (a) Assume first that P_0 is a point of a stable equilibrium in an electrostatic field. What is the force acting on a charge in equilibrium at the point P_0? How does the electric field act around P_0?

   (b) Consider now a Gaussian surface at a small distance around P_0. Apply Gauss' law and conclude that there must be a charge inside the considered volume if we assume P_0 to be a stable equilibrium point.

   (c) What can you conclude about the equilibrium in the electrostatic fields?

2. Consider a conductor with a cavity. There is no electric field in the metal, but what about in the cavity?

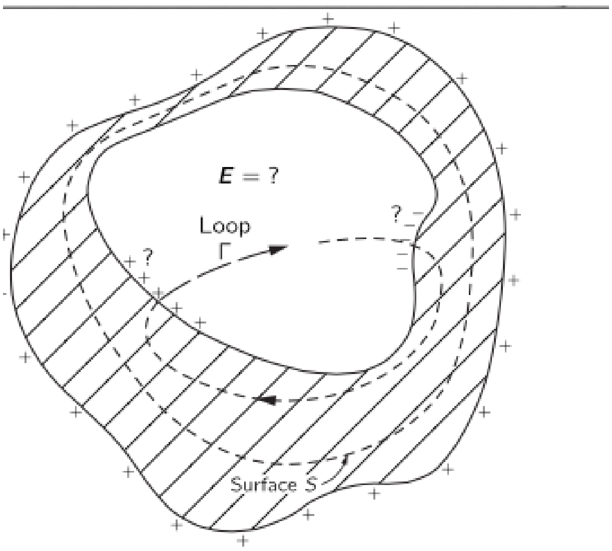


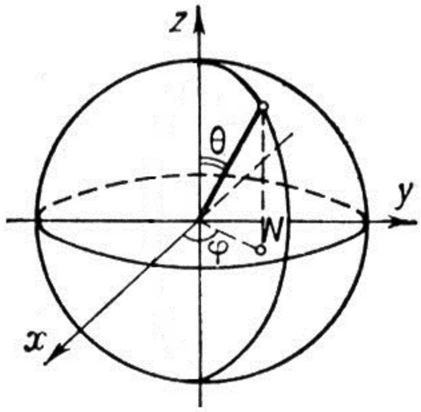
Figure 1: *Cavity*

   (a) Let us consider a cavity of a conductor (for any shape) like the one given in Fig.1. Consider a Gaussian surface, *e.g.* S in the figure, that encloses the cavity but stays everywhere in the conducting material. Could there be a positive surface charge on one part and a negative one somewhere else, as indicated in the figure?

   (b) Now imagine a loop that crosses the cavity along a line of force from some positive to some negative charge, and returns to its starting point via the conductor (see Fig.1). What can you conclude by applying Stoke's theorem?

Hint: This exercise is based on the Feynman lectures Vol. 2, Ch.05-4. You can find the answer in the link below, but it will be good to test your ability to answer yourselves first.
www.feynmanlectures.caltech.edu

Exercise 2. ***Electric field of a sphere***



Consider a spherical shell of radius R_0, charged with an electric charge Q which is uniformly distributed.

a) Write down the charge density $\rho(\mathbf{x})$ of the shell.

Solution. In spherical coordinates, the charge density is of the form

$$\rho(\mathbf{x}(r,\theta,\phi)) = C\,\delta(r - R_0)\,.$$

We determine the constant C by integrating over the whole sphere:

$$Q \stackrel{!}{=} \int_V \mathrm{d}^3x \rho(\mathbf{x}) = C \int_0^\infty \mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi \int_0^\pi \mathrm{d}\theta r^2 \sin(\theta)\delta(r - R_0) = C 4\pi R_0^2\,,$$

so $C = Q/(4\pi R_0^2)$, the total charge divided by the surface area of the spherical shell.
We get the charge density as

$$\rho(\mathbf{x}(r,\theta,\phi)) = Q/(4\pi R_0^2)\,\delta(r - R_0) \equiv \sigma\,\delta(r - R_0)$$

with the surface charge density σ.

b) Using the symmetry of the system and Gauss' law, find the electric field $\mathbf{E}$ inside and outside of the shell.

Solution. The charge distribution is *spherically* symmetric (i.e. it only depends on the radial variable r). Therefore, the electric field $\mathbf{E}$ both inside and outside of the shell can only depend on r:

$$\mathbf{E}(\mathbf{x}) = E(r)\,\mathbf{e}_r\,.$$

Gauss' law reads

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\epsilon_0}\,. \tag{S.1}$$

The volume integral of the left hand side can be written as a surface integral, using the divergence theorem:

$$\int_V \mathrm{d}^3x \nabla \cdot \mathbf{E}(\mathbf{x}) = \int_{\partial V} \mathbf{E} \cdot \mathrm{d}\mathbf{A}$$

If we choose the volume V to be a sphere of radius R, the electric field is constant on the surface and we simply get

$$\int_V \mathrm{d}^3x \nabla \cdot \mathbf{E}(\mathbf{x}) = 4\pi R^2 E(R) \, .$$

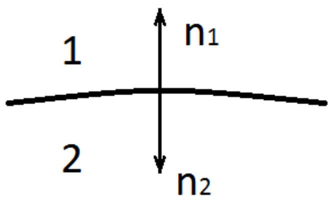
The integral of the RHS of eq. S.1 is simply the total charge contained in the integration volume, divided by ϵ_0. Thus, we find the result for the electric field:

$$E(r) = \begin{cases} 0 & , r < R_0 \\ \frac{Q}{4\pi\epsilon_0 r^2} & , r > R_0 \end{cases} \, .$$

c) Consider an element of arbitrary surface with charge density ρ shown in the figure. Show that the electric field due to the charges external to this surface element can be calculated as

$$\mathbf{E}_{ext} = \frac{\mathbf{E_1} + \mathbf{E_2}}{2} \, , \tag{1}$$

where $\mathbf{E_1}$ and $\mathbf{E_2}$ correspond to the electric fields in regions 1 and 2.



Solution. The electric field created by the element itself is $\mathbf{E}_{own} = 2\pi\sigma\mathbf{n}\frac{1}{4\pi\epsilon_0}$ (This can be easily obtained from the divergence theorem with an elementary box as a surface). Thus, the electric field at any point of the subspace marked with 1 near the surface can be calculated as

$$\mathbf{E}_1 = \mathbf{E}_{ext} + 2\pi\sigma\mathbf{n}_1\frac{1}{4\pi\epsilon_0} \, . \tag{S.2}$$

Same for the subspace 2:

$$\mathbf{E}_2 = \mathbf{E}_{ext} + 2\pi\sigma\mathbf{n}_2\frac{1}{4\pi\epsilon_0} \, . \tag{S.3}$$

Since the two unit vectors have same magnitude and opposite directions adding the two equations together we obtain the required equality.
In the special case of the spherical shell, $\mathbf{E}_2 = 0$ as we have seen in part b), so the "external field" that a infinitesimal surface area feels is exactly half the field outside of the sphere.

d) Now, consider the sphere of parts a) and b) to be split into two hemispheres by the plane $z = 0$. Using the results from all the previous parts, calculate the force with which the two hemispheres repel each other.

Solution.

- The force acting on an element of a surface can only be caused by the charges external to that element and can be calculated as (per unit surface) $\mathbf{F} = \sigma \mathbf{E}_{ext}$.
- Due to symmetry this force has to be perpendicular to the surface of the sphere at any point. Using the field inside and outside the sphere calculated in part b), as well as part c)'s result, we get $|\mathbf{F}| = \sigma \frac{Q}{2R_0^2} \frac{1}{4\pi\epsilon_0}$.
- Given the way we divided the sphere it is clear from symmetry that the resulting force will be along z axis, so we will have to sum up only the vertical contribution of all elementary forces $F_z = \sigma \frac{Q}{2R_0^2} \frac{1}{4\pi\epsilon_0} cos(\theta)$. (One can easily check that the x- and y-component of the force integrate to zero).
- The surface element at constant radius R_0 is $\mathrm{d}S = R_0^2 sin(\theta) \mathrm{d}\theta \mathrm{d}\phi$ and the charge density found in part a) is $\sigma = \frac{Q}{4\pi R_0^2}$.
- The magnitude of the total force felt by the upper hemisphere is:

$$\begin{aligned} F_z &= \int F_z dS = \int_0^{\pi/2} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \frac{\sigma Q sin(\theta) cos(\theta)}{2} \frac{1}{4\pi\epsilon_0} \\ &= \frac{\sigma Q \pi}{2} \frac{1}{4\pi\epsilon_0} = \frac{Q^2}{8R_0^2} \frac{1}{4\pi\epsilon_0} , \end{aligned} \tag{S.4}$$

  and the force on the lower hemisphere is the same, just in the opposite direction.

Exercise 3. *Energy of a charge distribution*

Find the energy stored in two different configurations:

1. a charged spherical conductor
2. a uniformly charged solid sphere

both with radius R and charge Q. Explain the energy difference between the two cases.

Solution. First, we consider the uniformly charged sphere; we have that the uniform charge density in the sphere is just

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{3Q}{4\pi R^3}.$$

Due to symmetry, the electric field will just be $\mathbf{E} = E(r)\,\hat{\mathbf{r}}$; we can evaluate it by scratch with Gauss' theorem.

For $r < R$, we have

$$E(r)\,4\pi r^2 = \rho \frac{4}{3}\pi r^3 \frac{1}{\epsilon_0} \;\Rightarrow\; E(r) = \frac{\rho r}{3\epsilon_0},\; r < R;$$

for $r > R$, the field is just the Coulomb field

$$E(r) = \frac{\rho R^3}{3r^2\epsilon_0}.$$

Ve can now evaluate the potential Φ by imposing the correct boundary condition, i.e.

- $\Phi(r) \to 0$ as $r \to \infty$;
- $\Phi(r)$ continuous at $r = R$.

and recalling that, in our case,

$$\mathbf{E} = -\nabla\Phi = -\frac{\partial}{\partial r}\Phi\,\hat{\mathbf{r}}.$$

The right solution is then

$$\Phi(r) = \begin{cases} \frac{\rho}{6\epsilon_0}(3R^2 - r^2) & r \leq R \\ \frac{\rho R^3}{3r\epsilon_0}, & r > R. \end{cases}$$

Then, we can evaluate the energy with $\mathbf{E}$ in the two different ways. First, with the formula

$$W = \frac{\epsilon_0}{2}\left(\int_V |\mathbf{E}(\mathbf{x})|^2 \mathrm{d}^3x + \oint_{\partial V} \Phi(\mathbf{x})\mathbf{E}(\mathbf{x}) \cdot \mathrm{d}\mathbf{A}\right)$$

Take a sphere of radius $X > R$. The integral we want to evaluate is the sum of three contributions: the integral over the charged sphere, the integral between R and X, and the surface integral.

$$\mathcal{I} = 2\pi\epsilon_0\left\{\int_0^R \mathrm{d}r\, r^2\left(\frac{\rho}{3\epsilon_0}r\right)^2 + \int_R^X \mathrm{d}r\, r^2\left(\frac{\rho R^3}{3r^2\epsilon_0}\right)^2 + \int_{\partial V, r=X} \frac{\rho R^3}{3r\epsilon_0}\frac{\rho R^3}{3r^2\epsilon_0}\,\mathrm{d}\sigma\right\},$$

where the surface integral is easily computed, since ∂V is just a sphere of surface $4\pi r^2$ and the integrand does *not* depend on the angles; the other integrals are straightforward. All in all, we have

$$W = \frac{2\pi}{\epsilon_0}\left\{\frac{\rho^2 R^5}{45} + \frac{\rho^2 R^6}{9}\left[-\frac{1}{r}\right]_R^X + \frac{\rho^2 R^6}{9X}\right\} = \frac{4\pi}{15\epsilon_0}R^5\rho^2.$$

Of course, we can evaluate the third point by just taking the limit $X \to \infty$, since we assume that the potential drops at infinity; of course, the result is the same.

If we now evaluate the energy with the formula $W = \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x})\mathrm{d}^3x$, we get

$$W = 2\pi \int_0^R \mathrm{d}r\, r^2 \frac{\rho^2}{6\epsilon_0}(3R^2 - r^2) = \frac{2\pi}{\epsilon_0}\left[\frac{\rho^2}{6}\left(R^5 - \frac{R^5}{5}\right)\right] = \frac{4\pi}{15\epsilon_0}R^5\rho^2.$$

Replacing the charge density ρ we obtain for the energy:

$$\boxed{W = \frac{1}{4\pi}\frac{3Q^2}{5R\epsilon_0}}.$$

For the charged shell, we can write the charge density as

$$\rho(r) = \frac{Q}{4\pi R^2}\delta(r - R).$$

We get that the field now as

$$E(r) = \begin{cases} 0, & r < R \\ \frac{Q}{(4\pi r^2\epsilon_0)} & r > R. \end{cases}$$

By the same arguments of continuity as before, the potential is

$$\Phi(r) = \begin{cases} \frac{Q}{(4\pi R\epsilon_0)} & r < R \\ \frac{Q}{(4\pi r\epsilon_0)}, & r > R. \end{cases}$$

and the integrals for the energy are similar.

With the formula $W = \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x})\mathrm{d}^3x$ we have:

$$W = 2\pi \int_0^\infty \mathrm{d}r\, r^2 \frac{Q}{4\pi R^2}\frac{Q}{(4\pi r\epsilon_0)}\delta(r - R) = \boxed{\frac{1}{4\pi\epsilon_0}\frac{Q^2}{2R}}.$$

Then we have that $\mathbf{E}$ inside the sphere is zero, thus the second formula $W = \frac{\epsilon_0}{2}\left(\int_V |\mathbf{E}(\mathbf{x})|^2 \mathrm{d}^3x + \oint_{\partial V} \Phi(\mathbf{x})\mathbf{E}(\mathbf{x}) \cdot \mathrm{d}\mathbf{A}\right)$ leads us to:

$$\begin{aligned} W =& 2\pi\epsilon_0 \int_R^X \mathrm{d}r\, r^2\left(\frac{Q}{4\pi r^2\epsilon_0}\right)^2 + \frac{1}{2}\int_{\partial V} \frac{Q}{(4\pi r\epsilon_0)}\frac{Q}{(4\pi r^2\epsilon_0)}\,\mathrm{d}\sigma \\ =& \frac{2\pi}{\epsilon_0}\frac{Q^2}{16\pi^2}\left[-\frac{1}{r}\right]_R^X + \frac{2\pi}{\epsilon_0}\frac{Q^2}{16\pi^2 X} = \frac{Q^2}{8\pi R\epsilon_0}, \end{aligned}$$

which of course concides with our previous result for the charged shell.

The difference in energy between the two different configurations lies entirely inside the sphere. In the first case, the electric field inside the sphere grows linearly, whereas in the latter case is zero. Therefore, the second configuration lacks the contribution to the energy due to the electric field *inside* the sphere.

Exercise 4. *The Electrostatic Potential of NaCl*

In this exercise we will calculate the energy per molecule required to separate the ionic crystal NaCl (salt) into its ions.

The NaCl crystal consists of negative and positive ions whose structure we know from x-ray diffraction. It is a cubic lattice and the cross section of the crystal is depicted in Fig.2. We assume that the ions Na^+ and Cl^- are of the same size. The energy of this lattice originates from the electrical interaction of the atoms, a repulsion which becomes important if ions are compressed too close together, and a kinetic energy due to vibrations. In this problem we will ignore the repulsive force and the lattice vibrations and approximate the required energy to pull apart the NaCl ions as the electrostatic potential between the ions.

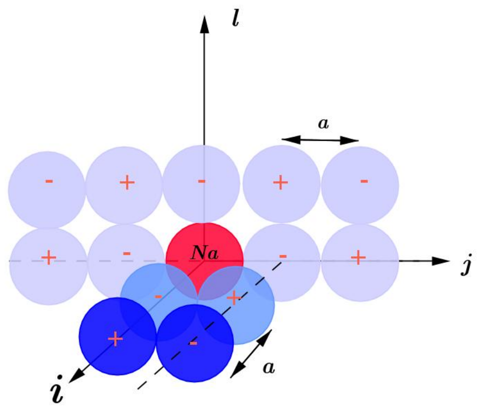


Figure 2: *NaCl*

1. Consider only one ion assigned as Na^+ in the figure. Write down the energy due to the electrostatic interaction of this ion and the neighbours in the horizontal line along the $i-$axis. The distance between the Na^+ ion assigned in the Figure and the nearest Cl^- is $a = 2.81\,10^{-8}$ cm. Evaluate the sum analytically.

   Solution. Let the distance between the Na^+ ion assigned in the Fig.2, and the nearest Cl^- ion a. The energy between these two is e^2/a. Next to these Cl^- there are the positive ions at a distance of $2a$ from the original ion, and so on. Taking into account all the contributions in the horizontal line we find:

$$U_1 = \frac{e^2}{a}\left(-\frac{2}{1}+\frac{2}{2}-\frac{2}{3}+\frac{2}{4}\mp\cdots\right) \tag{S.5}$$

$$= -\frac{2e^2}{a}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\pm\cdots\right) \tag{S.6}$$

$$= -\frac{2e^2}{a}\log 2 \tag{S.7}$$

2. Now consider the nearest next line and write down the sum for the electrostatic potential. Note that there are four such lines surrounding the chosen ion.

   Solution. In the next line the nearest ion is at a distance a. The next ions are positive and at a distance $\sqrt{2}a$, and so on. Taking into account that there are 4 lines and summing them all, we obtain:

$$U_2 = \frac{e^2}{a}\left(-\frac{1}{1}+\frac{2}{\sqrt{2}}-\frac{2}{\sqrt{5}}+\frac{2}{\sqrt{10}}\mp\cdots\right) \tag{S.8}$$

3. Write down an expression taking into account the rest of the lines. Evaluate your result numerically (in the language of your choice). Now compare your result to the experimentally measured result of 7.92 eV per molecule. Are they of the same order of magnitude? How would you interpret the small discrepancy?

Solution. The expression that takes into account the contribution of all ions is:

$$U = U_1+U_2+U_3 = -2\frac{e^2}{a}\log(2) - 4\frac{e^2}{a}\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{j+l+1}}{\sqrt{j^2+l^2}} - 4\frac{2e^2}{a}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+j+l+1}}{\sqrt{j^2+l^2+i^2}} \approx -1.747\frac{e^2}{a} = -8.94\,\text{eV}. \tag{S.9}$$

Where we used $\frac{e^2}{a} = 5.12$ eV and

$$U_1 = -2\frac{e^2}{a}\log(2) \tag{S.10}$$

is the contribution along the i-axis,

$$U_2 = -4\frac{e^2}{a}\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{j+l+1}}{\sqrt{j^2+l^2}} \tag{S.11}$$

is the energy due to all the neighbour in the $j-l$-plane, and

$$U_3 = -4\frac{2e^2}{a}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+j+l+1}}{\sqrt{j^2+l^2+i^2}} \tag{S.12}$$

is the one due to the remaining ions (everything except the i-axis and the $j-l$-plane).