Exercise 1. ***Vector Identities.***

In Electrodynamics we frequently use standard vector identities. To practice with Einstein summation convention prove the following identities:

1. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}$
2. $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})\, (\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})\, (\mathbf{b} \cdot \mathbf{c})$
3. $\mathbf{Ra} \times \mathbf{Rb} = \mathbf{R}\, (\mathbf{a} \times \mathbf{b})$
4. $\nabla \times \nabla\psi = 0$
5. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
6. $\nabla \times (\nabla \times \mathbf{A}) = \nabla\, (\nabla \cdot \mathbf{A}) - \Delta\mathbf{A}$
7. $\nabla\, (\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\, \mathbf{B} + (\mathbf{B} \cdot \nabla)\, \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are vectors, $\mathbf{A}$, $\mathbf{B}$ are vectorfields, ψ is a function and $\mathbf{R} \in \mathrm{SO}(3)$. Moreover assume that all components A_i, B_j and also ψ are in $\mathcal{C}(2)$, i.e. two times continuously differentiable.
Don't write out cross products explicitly, but use the index notation involving the Levi-Civita symbol ε_{ijk}.

Solution.

1.

$$\begin{aligned}
(\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))_i &= \varepsilon_{ijk} a_j \varepsilon_{klm} b_l c_m \\
&= \varepsilon_{kij} \varepsilon_{klm} a_j b_l c_m \\
&= a_j b_i c_j - a_j b_j c_i \\
&= (\mathbf{a} \cdot \mathbf{c}) b_i - (\mathbf{a} \cdot \mathbf{b}) c_i
\end{aligned}$$

where we used $\varepsilon_{ijk} = \varepsilon_{kij}$ and $\varepsilon_{kij}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

2.

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= \varepsilon_{ijk} a_j b_k \varepsilon_{iml} c_m d_l \\
&= a_j b_k c_j d_k - a_j b_k c_k d_j \\
&= (\mathbf{a} \cdot \mathbf{c})\, (\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})\, (\mathbf{b} \cdot \mathbf{c})
\end{aligned}$$

where we used $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{il}$

3. First, in the spirit of the expansion of the determinant of a matrix M we observe

$$\varepsilon_{jml} M_{ji} M_{mn} M_{ls} = \varepsilon_{ins} \det(M) \qquad . \tag{S.1}$$

Hence, we find with $R^{-1} = R^T$ and $\det(R) = 1$

$$\begin{aligned}
\left(R^{-1}\, (R\mathbf{a} \times R\mathbf{b})\right)_i &= R_{ji}\, (R\mathbf{a} \times R\mathbf{b})_j \\
&= R_{ji} \varepsilon_{jml} R_{mn} a_n R_{ls} b_s \\
&= \varepsilon_{ins} a_n b_s \\
&= (\mathbf{a} \times \mathbf{b})_i
\end{aligned}$$

4.

$$(\nabla \times \nabla\psi)_i = \varepsilon_{ijk}\partial_j\partial_k\psi = 0$$

since the partial derivatives commute and $\varepsilon_{i,j,k}$ is antisymmetric.

5.

$$\nabla \cdot (\nabla \times \mathbf{A}) = \varepsilon_{ijk}\partial_i\partial_j\mathbf{A}_k = 0$$

since the partial derivatives commute and $\varepsilon_{i,j,k}$ is antisymmetric.

6.

$$\begin{aligned}
(\nabla \times (\nabla \times \mathbf{A}))_i &= \varepsilon_{ijk}\varepsilon_{klm}\partial_j\partial_l\mathbf{A}_m \\
&= \varepsilon_{kij}\varepsilon_{klm}\partial_j\partial_l\mathbf{A}_m \\
&= \partial_i\partial_m\mathbf{A}_m - \partial_j\partial_j\mathbf{A}_i \\
&= (\nabla(\nabla\cdot\mathbf{A}) - \Delta\mathbf{A})_i
\end{aligned}$$

where we used $\varepsilon_{ijk} = \varepsilon_{kij}$ and $\varepsilon_{kij}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

7.

$$\begin{aligned}
&((\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}))_i \\
&= \mathbf{A}_j\partial_j\mathbf{B}_i + \mathbf{B}_j\partial_j\mathbf{A}_i + \varepsilon_{ijk}\varepsilon_{klm}\mathbf{A}_j\partial_l\mathbf{B}_m + \varepsilon_{ijk}\varepsilon_{klm}\mathbf{B}_j\partial_l\mathbf{A}_m \\
&= \mathbf{A}_j\partial_j\mathbf{B}_i + \mathbf{B}_j\partial_j\mathbf{A}_i + \varepsilon_{kij}\varepsilon_{klm}\mathbf{A}_j\partial_l\mathbf{B}_m + \varepsilon_{kij}\varepsilon_{klm}\mathbf{B}_j\partial_l\mathbf{A}_m \\
&= \mathbf{A}_j\partial_j\mathbf{B}_i + \mathbf{B}_j\partial_j\mathbf{A}_i + \mathbf{A}_j\partial_i\mathbf{B}_j - \mathbf{A}_j\partial_j\mathbf{B}_i + \mathbf{B}_j\partial_i\mathbf{A}_j - \mathbf{B}_j\partial_j\mathbf{A}_i \\
&= \mathbf{A}_j\partial_i\mathbf{B}_j + \mathbf{B}_j\partial_i\mathbf{A}_j \\
&= \partial_i(\mathbf{A}_j\mathbf{B}_j) \\
&= (\nabla(\mathbf{A}\cdot\mathbf{B}))_i
\end{aligned}$$

where we used $\varepsilon_{ijk} = \varepsilon_{kij}$ and $\varepsilon_{kij}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

Exercise 2. *Gauss and Stokes theorems.*

1. Consider the vector field in $\mathbb{R}^3$ (in Cartesian coordinates)

$$\mathbf{V}(x,y,z) = (x\,y,\; z^2\,y^2,\; z^2 + y), \tag{1}$$

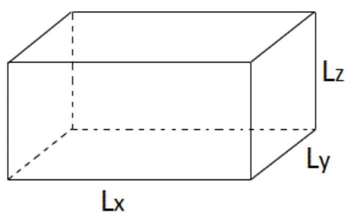
and a parallelepiped domain

$$\mathcal{D} = \{(x,y,z) \in \mathbb{R}^3 \,|\, 0 \le x \le L_x,\; 0 \le y \le L_y,\; 0 \le z \le L_z,\}. \tag{2}$$

Check the validity of the divergence theorem, by proving that

$$\int_{\mathcal{D}} \mathrm{d}^3x\, \nabla\cdot\mathbf{V} \;=\; \int_{\partial\mathcal{D}} \mathbf{V}\cdot\mathrm{d}\hat{\mathbf{A}}, \tag{3}$$

where $\partial\mathcal{D}$ is the border surface of the parallelepiped $\mathcal{D}$ in figure.



Note: Given a surface $\mathbf{A}$, parametrized as $\mathbf{A} = \{A_x(s,t), A_y(s,t), A_z(s,t)\}$, the surface vector $\mathrm{d}\hat{\mathbf{A}}$ is defined as

$$\mathrm{d}\hat{\mathbf{A}} = \frac{\partial \mathbf{A}}{\partial s} \times \frac{\partial \mathbf{A}}{\partial t}\, \mathrm{d}s\, \mathrm{d}t. \tag{4}$$

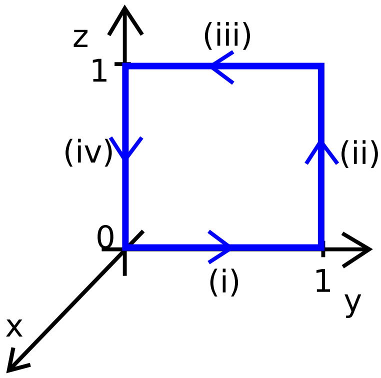
When parametrizing the parallelepiped, pay attention to the orientation of the surfaces.

2. Consider the vector field

$$\mathbf{V}(x,y,z) = \left(0, 6xz + 9y^2, 12yz^2\right)\ .$$

Check that $\mathbf{V}$ fulfills Stokes' theorem for the area/path defined in the figure, i.e. calculate both sides of the equation

$$\oint \mathbf{V} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{V}) \cdot d\hat{\mathbf{A}}\ . \tag{5}$$



Solution.

1. The divergence is

$$\nabla \cdot \mathbf{V} = y + 2yz^2 + 2z,$$

and the integral is easily evaluated as

$$\begin{aligned}\nabla \cdot \mathbf{V} &= \int_0^{L_x} \mathrm{d}x \int_0^{L_y} \mathrm{d}y \int_0^{L_z} \mathrm{d}z\, (y + 2yz^2 + 2z) \\ &= \frac{1}{2} L_x L_y^2 L_z + \frac{1}{3} L_x L_y^2 L_z^3 + L_x L_y L_z^2. \end{aligned} \tag{S.2}$$

For the surface integral, we evaluate separately the six integral on each surface, and we give them the correct orientation (so that the surface vector is pointing outside the parallelepiped).

$$\begin{aligned}
A_1 = (x, L_y - y, 0)\ \Rightarrow\quad & \textstyle\int_{A_1} \mathbf{V} \cdot \mathrm{d}\hat{\mathbf{A}} = \int_0^{L_x} \mathrm{d}x \int_0^{L_y} \mathrm{d}y\, (x\,y,\ z^2\,y^2,\ z^2 + y)|_{z=0} \cdot (0,0,-1) \\
& = -\tfrac{1}{2} L_x L_y^2 \\
A_2 = (x, y, L_z)\ \Rightarrow\quad & \textstyle\int_0^{L_x} \mathrm{d}x \int_0^{L_y} \mathrm{d}y\, (x\,y,\ z^2\,y^2,\ z^2 + y)|_{z=L_z} \cdot (0,0,1) \\
& = L_x L_y L_z^2 + \tfrac{1}{2} L_x L_y^2 \\
A_3 = (0, y, L_z - z)\ \Rightarrow\quad & \textstyle\int_0^{L_y} \mathrm{d}y \int_0^{L_z} \mathrm{d}z\, (x\,y,\ z^2\,y^2,\ z^2 + y)|_{x=0} \cdot (-1,0,0) \\
& = 0 \\
A_4 = (L_x, y, z)\ \Rightarrow\quad & \textstyle\int_0^{L_y} \mathrm{d}y \int_0^{L_z} \mathrm{d}z\, L_x y = \tfrac{1}{2} L_x L_y^2 L_z \\
A_5 = (x, 0, z)\ \Rightarrow\quad & \textstyle\int_0^{L_x} \mathrm{d}x \int_0^{L_z} \mathrm{d}z\, 0 = 0 \\
A_6 = (x, L_y, L_z - z \Rightarrow\quad & \textstyle\int_0^{L_x} \mathrm{d}x \int_0^{L_z} \mathrm{d}z\, L_y^2 z^2 = \tfrac{1}{3} L_x L_y^2 L_z^3,
\end{aligned}$$

and the sum is the same as eq. (S.2).

2. We have to prove Stokes' theorem:

$$\oint \mathbf{V} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{V}) d\hat{\mathbf{A}} \,.$$

To compute the right hand side of the equation (5) we need

$$\nabla \times \mathbf{V} = 6(2z^2 - x)\hat{\mathbf{x}} + 6z\hat{\mathbf{z}} \,.$$

We assume a counterclockwise line integral, therefore $d\hat{\mathbf{A}} = dydz\hat{\mathbf{x}}$. The surface integral is given by:

$$\int (\nabla \times \mathbf{V}) d\hat{\mathbf{A}} = \int_0^1 \int_0^1 12z^2 dydz = 4 \,. \tag{S.3}$$

For the left hand side of the equation (5) we have to evaluate the four line integrals:

$$\begin{array}{llll} \text{(i)} \quad x = 0, & z = 0, \quad \mathbf{V} \cdot d\mathbf{l} = 9y^2 dy, & \int \mathbf{V} \cdot d\mathbf{l} = 9 \int_0^1 y^2 dy = 3 \\ \text{(ii)} \quad x = 0, & y = 1, \quad \mathbf{V} \cdot d\mathbf{l} = 12z^2 dy, & \int \mathbf{V} \cdot d\mathbf{l} = 12 \int_0^1 z^2 dz = 4 \\ \text{(iii)} \quad x = 0, & z = 1, \quad \mathbf{V} \cdot d\mathbf{l} = 9y^2 dy, & \int \mathbf{V} \cdot d\mathbf{l} = 9 \int_1^0 y^2 dy = -3 \\ \text{(iv)} \quad x = 0, & y = 0, \quad \mathbf{V} \cdot d\mathbf{l} = 0, & \int \mathbf{V} \cdot d\mathbf{l} = \int_1^0 y^2 dz = 0 \,. \end{array}$$

So

$$\oint \mathbf{V} \cdot d\mathbf{l} = 3 + 4 - 3 + 0 = 4 \,.$$

Notice the step (iii): there is a temptation to write $d\mathbf{l} = -dy\hat{\mathbf{y}}$ here, since the path goes to the left. You can get away with this, if you insist, by running the integral from $0 \to 1$.

Exercise 3. *Electric field from a charged line.*

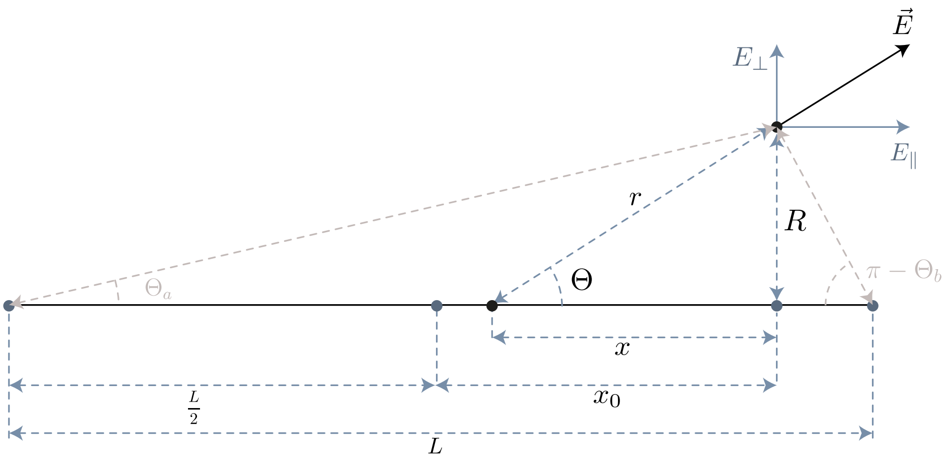
1. Consider an infinite line with constant charge density $\lambda = q/L$.
   (a) Using Gauss law, find the value of the electric field $\vec{E}$ generated by the line.
   (b) Compute $\vec{E}$ again, now using Coulomb's law.
   Hint. Find first the components of the electric field parallel ($E_{||}$) and perpendicular ($E_\perp$) to the line, as described in the picture.

2. Consider now a charged line of length L.
   (a) Compute the two components $E_{||}$ and $E_\perp$ of the electric field.
   Hint. Introduce a parameter x_0 related to a shift from the middle of the line. Be careful with the integration limits!
   (b) Take the limit for $L \to \infty$. You should recover the same values as in (1b).

3. Explore the limit $L/2 \ll (x_0, R)$. In this case the observation point is very far and we expect to recover the $1/r^2$ behavior of a point-like charge.

   Is it enough to expand up to first order, or do you need one more?

   Hint. Notice that now x_0 is not part of the line anymore, but is somewhere very far from it. Therefore the definition of the two angles θ_a and θ_b needs to be changed. Also, in order to compare the result with your expectation, write it in terms of radial and tangential components with respect to a spherical surface.



Solution.

1. (a) Thanks to the symmetries of the system, we can consider a cylinder of length L and radius R and use Gauss law

$$\begin{aligned} \int_S \vec{E}\cdot d\vec{S} &= \frac{1}{\epsilon_0}\int_V \rho(\vec{x})\,d\vec{x} \\ (2\pi R\,L)\,E &= \frac{1}{\epsilon_0}\,\lambda\,L \\ \Rightarrow E &= \frac{\lambda}{2\pi\epsilon_0}\,\frac{1}{R} \end{aligned} \tag{S.4}$$

where $\rho(\vec{x})$ is the charge density, here just λ, and $d\vec{S}$ the vector representing an infinitesimal surface element.

(b) The electric field of an infinitesimal line element is given by

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\hat{r}. \tag{S.5}$$

The infinitesimal charge element can be written as $dq = \lambda dx$. Then the total electric field is found by integrating $d\vec{E}$ from $-\infty$ to $+\infty$.

In order to evaluate the integral we introduce (according to the picture)

$$x = r\,\cos\theta \quad \text{and} \quad R = r\,\sin\theta, \tag{S.6}$$

such that the two components $E_\perp$ and $E_\parallel$ are given by

$$\begin{aligned} E_{\perp,\parallel} &= \int_{-\infty}^{+\infty} \frac{\lambda}{4\pi\epsilon_0}\,\frac{(\sin\theta,\,\cos\theta)}{(x^2+R^2)}dx \\ &= -\int_{\pi}^{0} \frac{d\theta}{\cos^2\theta}\,\frac{1}{R}\,\frac{R^2}{\tan^2\theta}\,\frac{\lambda}{4\pi\epsilon_0}\,\frac{(\sin\theta,\,\cos\theta)}{(R^2/(\tan^2\theta)+R^2)} \\ &= \frac{\lambda}{4\pi\epsilon_0}\,\frac{1}{R}\int_0^{\pi} d\theta(\sin\theta,\,\cos\theta) \\ &= \left(\frac{\lambda}{2\pi\epsilon_0}\,\frac{1}{R},\,0\right), \end{aligned} \tag{S.7}$$

where from the first to the second line we made the change of variables

$$\tan\theta = R/x, \qquad dx = -\frac{x^2}{R}\,\frac{d\theta}{\cos^2\theta}.$$

The result (S.7) is in agreement with part (1a).

2. The integrand is the same as in part (1b), what we have to be careful about are the integration limits:

$$\begin{aligned} E_{\perp,\,\parallel} &= \int_{-L/2}^{+L/2} \frac{\lambda}{4\pi\epsilon_0} \frac{(\sin\theta,\, \cos\theta)}{(x^2+R^2)} dx \\ &= \frac{\lambda}{4\pi\epsilon_0} \frac{1}{R} \int_{\theta_a}^{\theta_b} d\theta (\sin\theta,\, \cos\theta) \\ &= \frac{\lambda}{4\pi\epsilon_0} \frac{1}{R} \left(\cos\theta_a - \cos\theta_b,\, \sin\theta_b - \sin\theta_a\right), \end{aligned} \tag{S.8}$$

where θ_a and θ_b are the two angles related to the left and right boundaries of the line. As displayed in the picture they are given by

$$\theta_a = \arctan\left(\frac{R}{x_0 + L/2}\right) \tag{S.9}$$

$$\pi - \theta_b = \arctan\left(\frac{R}{L/2 - x_0}\right), \tag{S.10}$$

where x_0 refers to the displacement from the middle of the line. Plugging θ_a and θ_b into Eq. (S.8) leads to

$$\begin{aligned} E_\perp &= \frac{\lambda}{4\pi\epsilon_0 R}\left[\cos\left(\arctan\left(\frac{R}{x_0+L/2}\right)\right) - \cos\left(\pi - \arctan\left(\frac{R}{L/2-x_0}\right)\right)\right] \\ &= \frac{\lambda}{4\pi\epsilon_0 R}\left[\cos\left(\arctan\left(\frac{R}{x_0+L/2}\right)\right) + \cos\left(\arctan\left(\frac{R}{L/2-x_0}\right)\right)\right] \end{aligned} \tag{S.11}$$

$$\begin{aligned} E_\parallel &= \frac{\lambda}{4\pi\epsilon_0 R}\left[\sin\left(\pi - \arctan\left(\frac{R}{L/2-x_0}\right)\right) - \sin\left(\arctan\left(\frac{R}{x_0+L/2}\right)\right)\right] \\ &= \frac{\lambda}{4\pi\epsilon_0 R}\left[\sin\left(\arctan\left(\frac{R}{L/2-x_0}\right)\right) - \sin\left(\arctan\left(\frac{R}{x_0+L/2}\right)\right)\right] \end{aligned} \tag{S.12}$$

3. We start by noting that for $x_0 > L/2$ the two angles (S.9) and (S.10) are given by

$$\theta_a = \arctan\left(\frac{R}{x_0 + L/2}\right) \tag{S.13}$$

$$\theta_b = \arctan\left(\frac{R}{x_0 - L/2}\right). \tag{S.14}$$

The two components $E_\perp$ and $E_\parallel$ now become

$$\begin{aligned} E_{\perp,\parallel} = \frac{\lambda}{4\pi\epsilon_0 R}\Big[&\cos\left(\arctan\left(\frac{R}{x_0+L/2}\right)\right) - \cos\left(\arctan\left(\frac{R}{x_0-L/2}\right)\right), \\ &\sin\left(\arctan\left(\frac{R}{x_0-L/2}\right)\right) - \sin\left(\arctan\left(\frac{R}{x_0+L/2}\right)\right)\Big]. \end{aligned} \tag{S.15}$$

Let us define $r^2 = R^2 + x_0^2$ and let us now expand the terms appearing in Eq. (S.15) in $\frac{L}{r}$. We first approximate

$$\begin{aligned} \sin\left(\arctan\left(\frac{R}{x_0 \pm \frac{L}{2}}\right)\right) &= \frac{R}{\sqrt{\left(x_0 \pm \frac{L}{2}\right)^2 + R^2}} \\ &= \frac{R/r}{\sqrt{1 \pm \frac{x_0 L}{r^2} + \frac{L^2}{4r^2}}} \\ &\approx \frac{R}{r}\left(1 \mp \frac{x_0 L}{2r^2}\right), \end{aligned} \tag{S.16}$$

and

$$\begin{aligned} \cos\left(\arctan\left(\frac{R}{x_0 \pm \frac{L}{2}}\right)\right) &= \frac{x_0 \pm \frac{L}{2}}{R} \sin\left(\arctan\left(\frac{R}{x_0 \pm \frac{L}{2}}\right)\right) \\ &\approx \frac{x_0 \pm \frac{L}{2}}{R} \frac{R}{r}\left(1 \mp \frac{x_0 L}{2r^2}\right) \\ &\approx \frac{x_0}{r} \pm \frac{R^2 L}{2r^3}. \end{aligned} \tag{S.17}$$

Then, using these expansions we get for the electric field components

$$\begin{aligned}
E_\perp &\approx \frac{\lambda}{4\pi\epsilon_0 R}\left(\left(\frac{x_0}{r}+\frac{R^2L}{2r^3}\right)-\left(\frac{x_0}{r}-\frac{R^2L}{2r^3}\right)\right)\\
&= \frac{\lambda}{4\pi\epsilon_0 r^2}\frac{R}{r} = \frac{\lambda}{4\pi\epsilon_0 r^2}\sin\Theta, \qquad \text{(S.18)}\\
E_\parallel &\approx \frac{\lambda}{4\pi\epsilon_0 R}\left(\left(\frac{R}{r}+\frac{x_0RL}{2r^3}\right)-\left(\frac{R}{r}-\frac{x_0RL}{2r^3}\right)\right)\\
&= \frac{\lambda}{4\pi\epsilon_0 r^2}\frac{x_0}{r} = \frac{\lambda}{4\pi\epsilon_0 r^2}\cos\Theta \qquad \text{(S.19)}
\end{aligned}$$

At the end we find the following expressions for the radial and tangential parts of the electric field

$$\begin{aligned}
E_r &= E_\perp \sin\Theta + E_\parallel \cos\Theta\\
&\approx \frac{\lambda L}{4\pi\epsilon_0 r^2}\left(\cos^2\Theta+\sin^2\Theta\right)\\
&= \frac{Q}{4\pi\epsilon_0 r^2}, \qquad \text{(S.20)}
\end{aligned}$$

and

$$\begin{aligned}
E_t &= E_\parallel \sin\Theta - E_\perp \cos\Theta\\
&\approx \frac{\lambda L}{4\pi\epsilon_0 r^2}\left(\cos\Theta\sin\Theta-\sin\Theta\cos\Theta\right)\\
&= 0. \qquad \text{(S.21)}
\end{aligned}$$

We thus see that the expected $1/r^2$ behavior for a point-like charge is recovered! Note that by simply taking into account the first order terms we obtain zero field both in radial and tangential directions.