Exercise 1. ***Group velocity***

A Gaussian wave package $u(x,t)$ is moving in a dispersive Medium, i.e. ω does not depend linearly on k. At the time $t = 0$ we have

$$u(x, t = 0) = e^{-\frac{x^2}{2(\Delta x)^2}}, \tag{1}$$

where Δx can be interpret as a measure for the uncertainty. The time dependence is given by

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk u(k) e^{i(kx - \omega(k)t)} \tag{2}$$

where $u(k)$ is the Fourier transform of $u(x, t = 0)$.

a) Show by completing the square that the wave package in the momentum space also has a Gaussian profile (put $t = 0$). Is there any relation between Δx and Δk? What is the meaning for that?

b) Let k_0 be the vector for which $u(k_0)$ is maximal. Expand $\omega(k)$ at first order in k around k_0, put it into (2) and show that the maximum of the wave package (up to a phase factor) in the time is moved away by a factor of $v_g\, t$ from the Origin, where the group velocity is given by

$$v_g = \frac{d\omega}{dk}\,|_{k_0}\ . \tag{3}$$

c) Give the velocity of propagation for the single phases.

d) Gives an approximation of how fast is the dispersion of the wave package by finding an expression for the variation of the group velocity within the pulses. Interpret the result! Hint: use the result of part a).

Exercise 2. ***Reflection and Transmission***

Two mutually parallel planar surfaces, that are in the same, homogeneous, not magnetised and lossless dielectric with refractive index n, are separated by air ($n = 1$) on the distance d. From the upper half-space, the electromagnetic wave of frequency ω arrives with the angle α to the separating interface. Answer the following questions for the two cases - when the incoming wave is polarised parallel, and perpendicular to the plane of incidence:

a) Calculate the ratio of the reflected wave and that of the wave transmitted through the air to the lower half-space, to the incident wave.

b) Sketch for incident angles α larger than the critical angle for the total reflection, the ratio of the transmitted to incident wave as a function of the layer thickness d measured in the units of the wavelength.

Exercise 3. ***Photonic Band Gap Material***

Photonic Band Gap Materials are crystals in which some frequencies of light cannot propagate. The term *band gap* reminds one of the semiconductors, where electrons below certain energies are also not free. In this exercise, you will see a simple model of a Photonic Band Gap material.

a) Derive the generalised wave equations satisfied by the electric field $\vec{E}(\vec{r}, t)$ in a non-magnetic matter when the permitivity is a function of position, $\epsilon = \epsilon(\vec{r})$. Later on specialize to the case when $\epsilon(\vec{r}) = \epsilon(z)$ and also $\vec{E}(\vec{r}, t) = \hat{x}E(z, t)$.

b) Let $E(z, t) = E(z)exp(-i\omega t)$ and also let $\epsilon(z) = \epsilon_0[1 + \alpha \cos(2k_0 z)]$. Show that the Fourier components $\hat{E}(k)$ of the electric field satify the coupled set of the linear equations:

$$\left(k^2 - \frac{\omega^2}{c^2}\right) \hat{E}(k) = \frac{\omega^2 \alpha}{2c^2} \left[\hat{E}(k - 2k_0) + \hat{E}(k + 2k_0)\right]. \tag{4}$$

c) Suppose $\alpha << 1$ and focus on values of k in the vicinity of k_0, i.e. set $k = k_0 + q$, where $|q| << k_0$. Show that the Fourier components $\hat{E}(q + k_0)$ and $\hat{E}(q - k_0)$ are larger than all the others and therefore the 2×2 eigenvalue problem determines the dispersion relation. *Hint*: The wave frequency cannot differ greatly from its $\alpha = 0$ value in the limit considered.

d) Solve the eigenvalue problem to find $\omega(k_0, q)$. Study it's behavior at $q = 0$ (also remember that α is a small parameter.). Sketch the complete dispersion curve and show that there is a range of frequencies - called a photonic band gap - where no waves occur.