Exercise 1. ***Getting familiar with four-vectors***

In this exercise, you will become more familiar with the four-vector manipulations.
The greek indices μ, ν, ... take values 0, 1, ..., d, where d is the dimension of space (we are used to it being equal to 3, but it can be kept general).

1. **Derivative of a position vector:** Let now $x^\mu = (x^0,\ x^1,\ ...,\ x^d)$ and $\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial x^0},\ \frac{\partial}{\partial x^1},\ ...)$. What is $\partial_\mu x^\mu$? Can you see that it is indeed a Lorentz scalar?

2. **Lorentz tensors:** A general tensor can be written as an object with multiple indices, both up and down, i.e. $A^{\mu\nu\rho...}_{\gamma\delta\sigma...}$. Its transformation properties follow from the transformation properties of the tensor product of vectors, i.e. $x'^\mu y'^\nu = \Lambda^\mu_\sigma \Lambda^\nu_\gamma x^\sigma y^\gamma$ implies that $A'^{\mu\nu} = \Lambda^\mu_\sigma \Lambda^\nu_\gamma A^{\sigma\gamma}$.
   Prove however, that not every tensor can be written as a product of vectors, that is e.g. argue that it is not always possible to find $a^\mu,\ b^\nu$ such, that $S^{\mu\nu} = a^\mu b^\nu$ (even if $S^{\mu\nu}$ is symmetric).

3. **Symmetric and antisymmetric tensors**: In the following, let $A^{\mu\nu}$ be an antisymmetric tensor, that is $A^{\mu\nu} = -A^{\nu\mu}$ and $S^{\mu\nu}$ be a symmetric tensor: $S^{\mu\nu} = S^{\nu\mu}$.

   (a) Show that the (anti)symmetry of the tensor is preserved by the Lorentz transformations.

   (b) Prove that $A^{\mu\nu} S_{\mu\nu} = 0$.

   Let us now introduce a concept of symmetrization and antisymmetrization of a 2-index tensor [1]. For an arbitrary tensor $C^{\mu\nu}$ introduce it's symmetrisation: $C^{(\mu\nu)} = \frac{1}{2}\left(C^{\mu\nu} + C^{\nu\mu}\right)$ and antisymmetrisation: $C^{[\mu\nu]} = \frac{1}{2}\left(C^{\mu\nu} - C^{\nu\mu}\right)$.

   (c) Show that a general rank-2 tensor can be uniquely decomposed into the symmetric and antisymmetric part: $C^{\mu\nu} = C^{(\mu\nu)} + C^{[\mu\nu]}$.

Exercise 2. ***(More) General Lorentz transformation***

One usually performs Lorentz boosts along a specific axis, say x or z. In this exercise, we will perform a more general one, with a velocity vector in a specific plane.
In the lab frame, you are moving with a velocity $\vec{v} = (v, 0, 0)$, that is with a velocity v along the x axis. Another object is moving with the velocity $\vec{u} = (u_x, u_y, 0)$ as measured in the same frame.
What should your velocity v be so that the velocity of the object along the y axis, as measured by you, is also u_y.
(NB: $v = 0$ is clearly an answer; you are supposed to find the non-trivial one).
Hint: Recall that the four-velocity of an object in its rest frame is $(c, \vec{0})$.

[1] This concept can be generalised to higher order tensors. We will however not need it here

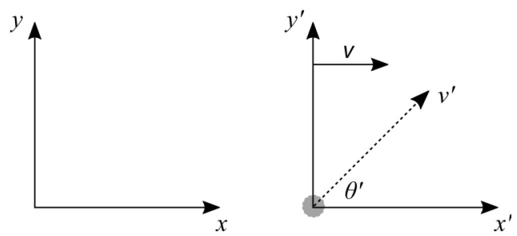
Exercise 3. ***Relativistic Doppler effect***

1. In 1929 Edwin Hubble proposed a law that was later named after him - that the galaxies seem to drift away from us the faster, the further away they already are:
$$v_{rec} = Hd \tag{1}$$
where v_{rec} is the Galaxy's receding velocity, d - it's distance from us and H is the Hubble's constant.
But how does one measure the velocity of the galaxy? One of the possibilities is to use the shift in the frequency of the emission lines of the elements constituting the matter of the galaxy. As hydrogen is the most common element, of special use is the *blue line of hydrogen* with the frequency of $\lambda = 434nm$ as measured in the laboratory on Earth (that is with the hydrogen atom at rest with respect to the observer).

   (a) Assume that in an astrophysical experiment a physicist detects a line of $\lambda' = 600nm$ in the spectrum of a distant galaxy. Assuming that it indeed corresponds to the blue line of hydrogen calculate the velocity with which the galaxy receds from the Earth.
   (b) The latest measure of the Hubble constant by the Planck experiment gave a result of $67.8 \frac{km}{s\ Mpc}$ [2]. For dimensional reasons, the inverse of the Hubble constant, *the Hubble time*, can be used as a rough estimate of the age of the Universe. See how good it is by comparing it with the most recent estimate of 13.7 billion years.

2. Consider now a frame $\mathcal{O}'$, moving away from the Earth (system $\mathcal{O}$), in which a light source is at rest. In this frame the light is emitted at an angle θ' with respect to the x'-axis and its frequency is ν'.

   (a) Assuming the Earth as fixed and the system $\mathcal{O}'$ moving with constant velocity v along the x-axis (i.e. x- and x'-axes are parallel, see figure below), find the observed angle θ from the x-axis in the system $\mathcal{O}$.
   (b) Assuming $v = 0.7c$, for which angle θ' does the Doppler shift vanish? What do you expect in the non-relativistic case ($v \ll c$)?



Exercise 4. ***Relativistic force***

Consider a constant force $F = const.$ acting on a free particle.

1. Give the expression of the velocity of the particle as a function of time and consider the limit $t \to \infty$.

2. Give the expression of the position as a function of time.

[2]Mpc - megaparsec. $1pc \approx 3.086 \times 10^{13} km$