Exercise 1. ***Spherical Hamonics***

In this exercise we want to become more confident with the Spherical Harmonics.

1. Starting from

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \, P_l^m(\cos(\theta)) \, e^{im\phi} \tag{1}$$

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l \tag{2}$$

with $x = \cos(\theta)$, derive the expressions for

$$Y_{00}, \, Y_{1,m}, \, Y_{2,m}, \qquad m \in [-l, l] \tag{3}$$

2. Draw the following functions

$$|Y_{00}|^2, \, |Y_{1,m}|^2, \, |Y_{2,m}|^2$$

for $m \in [-l, l]$ in a 3D plot (using for example Mathematica).

3. Verify the orthogonality conditions explicitly for the previous functions.
Notice that $Y_{l,m}(\theta, \phi)$ with m odd (even) are always odd (even) in θ.

Exercise 2. ***Spherical cavity and spherical functions***

Consider a sphere of radius a where the surface of the upper hemisphere has a potential $+\Phi_0$ and the surface of the lower hemisphere has a potential $-\Phi_0$, that is:

$$\Phi_0(\theta', \phi') = \begin{cases} +\Phi_0 & \text{for } \theta' \in [0, \frac{\pi}{2}] \\ -\Phi_0 & \text{for } \theta' \in (\frac{\pi}{2}, \pi]. \end{cases} \tag{4}$$

As you know from the lecture (method of image), in this case the Green Function is given by

$$G(\vec{r}, \vec{r}\,') = \frac{1}{|\vec{r} - \vec{r}\,'|} - \frac{a}{r' \left| \vec{r} - \frac{a^2}{r'^2} \vec{r}\,' \right|} \tag{5}$$

where $\vec{r}\,'$ refers to a unit source outside the sphere and $\vec{r}$ to the point where the potential is evaluated.

1. Using the expansion

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} \, Y_{lm}^*(\theta', \phi') \, Y_{lm}(\theta, \phi) \tag{6}$$

where $r_<(r_>)$ is the smaller (larger) of $|\vec{r}|$ and $|\vec{r}\,'|$, show that the Green Function (5) can be written as

$$G(\vec{r}, \vec{r}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a} \left(\frac{a^2}{r\, r'} \right)^{l+1} \right] Y_{lm}^*(\theta', \phi')\, Y_{lm}(\theta, \phi) \tag{7}$$

2. Using Dirichlet boundary conditions, show that the potential outside the sphere has following the expansion

$$\Phi(r, \theta, \phi) = \sum_{l,m} \left(\frac{a}{r} \right)^{l+1} Y_{lm}(\theta, \phi) \int \Phi_0(\theta', \phi') Y_{lm}^*(\theta', \phi') d\Omega'. \tag{8}$$

which tends to 0 as $r \to \infty$.

3. Now calculate the potential outside the sphere using (8) up to the terms of order a^4.

Hint. Notice that only terms with odd l will survive in the expansions (8) for the potential given by (4).

Exercise 3. ***Surface charge density***

As in the previous exercise, consider a spherical shell of radius R. The sphere has a charge density $\sigma(\theta)$ such that the surface potential has the following form

$$\Phi(R, \theta) = V_0 + V_1 \cos\theta + V_2 \cos 2\theta, \tag{9}$$

where V_0, V_1, V_2 are constants and θ is the polar angle.

1. Find the potential $\Phi(r, \theta)$ both inside and outside the spherical shell. Can be useful, in this case, to rewrite the potential in a unique form as

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left(a_l r^l + b_l r^{-(l+1)} \right) P_l(\cos\theta), \tag{10}$$

where P_l are the Legendre polynomials.

Hint. Use that $\Phi \to 0$ for $r \to \infty$ and the orthogonality of the Legendre polynomials.

2. Calculate the electric field $E(r, \theta)$.

3. Find the surface charge density $\sigma(\theta)$.

Hint. Use the fact that the component of the electric field orthogonal to the spherical surface undergoes a jump at $r = R$.

Exercise 4. ***Conductors and capacities***

In this problem we introduce and analyze the concept of capacity constants for arrays of conductors. Inside a conductor the electric field $\mathbf{E}$ vanishes and the electric potential is constant for static (i.e. equilibrium) situations. We consider finitely many perfect conductors described by spatially separated sets $A_1, \ldots, A_r$ for some $r \in \mathbb{N}$. Assume that they are carrying total charges $Q_1, \ldots, Q_r$ and that there exists a (unique) equilibrium charge density ρ (which of course vanishes outside the conductors).

1. The potential $V_i = V_i(\{Q_k\})$ is the potential of the $i-$th conductor. Show that

$$V_i(\{\lambda Q_k\}) = \lambda V_i(\{Q_k\}) \tag{11}$$

for any $\lambda \in \mathbb{R}$, using the explicit integral expression for a potential generated by a given charge distribution.

2. Using Eq. 11 show that $V_i = V_i(\{Q_k\})$ depends linearly on $Q_1, \ldots, Q_r$, i.e.

$$V_i(\{Q_k\}) = \sum_{j=1}^{r} D_{ij}\, Q_j, \tag{12}$$

where $D_{ik} := \frac{\partial V_i}{\partial Q_k}$ depends only on $\{A_k\}$.

It turns out that $D = (D_{ik})$ is regular. We define $C = (C_{ik}) := D^{-1}$. Its components C_{ik} are called capacity constants and depend only on the geometry $\{A_k\}$.

3. Show that the total energy W of the equilibrium charge distribution $\rho \equiv \rho(\{Q_k\}, \{A_k\})$ in the situation of the previous problem can be expressed as

$$W = \frac{1}{2} \sum_{i,j=1}^{r} C_{ij}\, V_i\, V_j. \tag{13}$$